Math 676: Day 2

The formal (& almost completely useless) definition of a manifold:

Def: A *differentiable manifold of dimension* n is a Hausdorff, second countable topological space M together with a family of injective maps $\varphi_\alpha : U_\alpha \subseteq \mathbb{R}^n \to M$ of open sets $U_\alpha \subseteq \mathbb{R}^n$ s.t.:

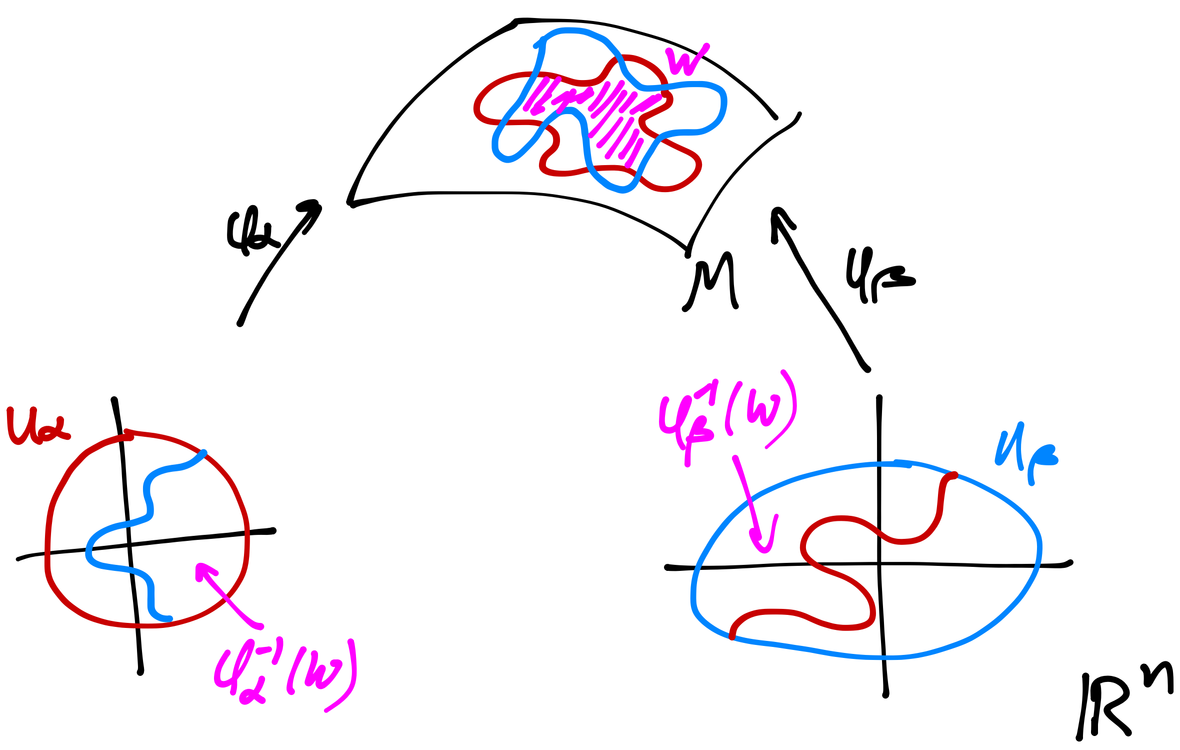
① $\bigcup_\alpha \varphi_\alpha(U_\alpha) = M$

② $\forall\, \alpha, \beta$ s.t. $\varphi_\alpha(U_\alpha) \cap \varphi_\beta(U_\beta) =: W \neq \emptyset$, the sets $\varphi_\alpha^{-1}(W)$ & $\varphi_\beta^{-1}(W)$ are open in $\mathbb{R}^n$ &

$$\varphi_\beta^{-1} \circ \varphi_\alpha,\ \varphi_\alpha^{-1} \circ \varphi_\beta : W \to \mathbb{R}^n \quad \text{are smooth } (C^\infty)$$

③ (Technical condition) the family $\{(U_\alpha, \varphi_\alpha)\}$ is maximal w.r.t. ① & ②.

The pairs $(U_\alpha, \varphi_\alpha)$ are called *local coordinate charts* for the manifold



Ex: • $\mathbb{R}^n$. with chart given by the identity map.
• S^n, the charts are given by inverse stereographic projection.

Def: Let M^m, N^n be mfds. A cont. map $f: M \to N$ is *differentiable* at $p \in M$ if, given a coord. chart $\psi : V \subseteq \mathbb{R}^n \to N$ containing $f(p)$, $\exists$ a chart $\varphi : U \subseteq \mathbb{R}^m \to M$ containing p s.t. $f(\varphi(U)) \subseteq \psi(V)$ &

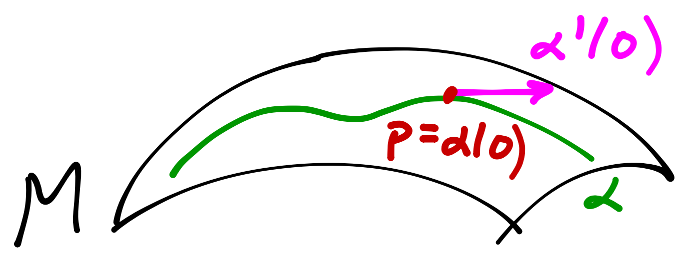
$$\psi^{-1} \circ f \circ \varphi : U \subseteq \mathbb{R}^m \to \mathbb{R}^n$$

is differentiable at $\varphi^{-1}(p)$.

Ex: $f: S^n \to S^n$ given by $f(\vec{x}) = -\vec{x}$.

Tangent Vectors

Intuition: Tangent vectors are derivatives of curves at points



It turns out that the way to formalize this is as a directional derivative operator.

Ex: Let $M=\mathbb{R}^n$ & let $\alpha:(-\varepsilon,\varepsilon)\to\mathbb{R}^n$ be smooth w/ $\alpha(0)=p$. Then we can write $\alpha(t)=(x_1(t),\dots,x_n(t))$ where $t\in(-\varepsilon,\varepsilon)$ & $x_1,\dots,x_n$ are smooth functions. Then $\alpha'(0)=(x_1'(0),\dots,x_n'(0))=:\vec{v}$ is a tangent vector at $p\in\mathbb{R}^n$.

If $f:\mathbb{R}^n\to\mathbb{R}$ is diff'ble in a nbhd of p, then the directional derivative of f at p in the direction of $\vec{v}$ is

$$\frac{d(f\circ\alpha)}{dt}\Big|_{t=0}=\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}\Big|_p\frac{dx_i}{dt}\Big|_{t=0}=\Big(\sum_i x_i'(0)\frac{\partial}{\partial x_i}\Big)f$$

Now, the thing inside the parentheses is a directional derivative operator acting on f which only depends on $\vec{v}$ & is a linear derivation: ① $\vec{v}(f+\lambda g)=v(f)+\lambda v(g)$ $\forall f,g$ diff'ble in a nbhd of p & $\lambda\in\mathbb{R}$

② $v(fg)=f(p)v(g)+g(p)v(f)$.

Df: Let M^n be a mfld. A smooth function $\alpha:(-\varepsilon,\varepsilon)\to M$ is a smooth curve in M. If $\alpha(0)=p\in M$ & $\mathcal{D}$ is the set of functions on M diff'ble at p, then the tangent vector to α at p is a functional

$$\alpha'(0):\mathcal{D}\to\mathbb{R}$$

given by $\alpha'(0)f=\dfrac{d(f\circ\alpha)}{dt}\Big|_{t=0}$

A tangent vector at p is the tangent vector at $t=0$ of some curve $\alpha:(-\varepsilon,\varepsilon)\to M$ w/ $\alpha(0)=p$.

The set of all tangent vectors at p is the tangent space T_pM.

For computations, usually need to write things in local coords. For example, let (U,φ) be a coord. chart containing $p\in M$ s.t. $\varphi(\bar{0})=p$. So for f diff'ble in a nbhd of p & $\alpha:(-\varepsilon,\varepsilon)\to M$ smooth w/ $\alpha(0)=p$,

$$\varphi^{-1}\circ\alpha(t)=(x_1(t),\dots,x_n(t))$$ for smooth functions $x_1,\dots,x_n$.

Then $\alpha'(0)f=\dfrac{d}{dt}(f\circ\alpha)\Big|_{t=0}=\dfrac{d}{dt}\big(f\circ\varphi(x_1(t),\dots,x_n(t))\big)\Big|_{t=0}=\sum_{i=1}^{n}x_i'(0)\dfrac{\partial f}{\partial x_i}=\Big(\sum x_i'(0)\dfrac{\partial}{\partial x_i}\Big|_{\bar{0}}\Big)f$

where I've abused notation to think of f as being a function of $(x_1,\dots,x_n)$; really, it's $f\circ\varphi$ that is a function of $(x_1,\dots,x_n)$.

This all means we can write the tangent vector $\alpha'(0) \in T_pM$ in local coords as

$$\alpha'(0) = \sum_i x_i'(0) \left(\frac{\partial}{\partial x_i}\right)_{\vec{0}}$$

Here the $\left(\frac{\partial}{\partial x_i}\right)_{\vec{0}}$ give the local coord. basis of T_pM associated to the chart (U, φ).

Different charts of course give different bases.

Def: the tangent bundle TM of a mfld M is the union $TM := \bigcup_{p \in M} T_pM$.

Likewise, if $(T_pM)^*$ is the dual of T_pM, then the cotangent bundle is the union of the cotangent spaces: $T^*M := \bigcup_{p \in M} (T_pM)^*$.

These are really bundles, meaning they come equipped w/ projections $\pi : TM \to M$, $(p, v) \mapsto p$ & $\tilde{\pi} : T^*M \to M$, $(p, \eta) \mapsto p$

Thm: If M is an n-dim'l mfld, then TM & T^*M are $2n$-dimensional manifolds.

In fact, T^*M is the classic example of a symplectic mfld (in physics terms, T^*M is position-momentum space, or phase space)