Math 676: Day 19

Remember that part of the definition of a moment map $\mu: M \to \mathfrak{g}^*$ was that it was equivariant w.r.t. the G action on (M,ω) and the coadjoint action of G on $\mathfrak{g}^*$, which we haven't defined yet...

Suppose G is a Lie gp w/ Lie algebra $\mathfrak{g}$, which we will identify w/ T_eG. For $g \in G$, let $L_g, R_g : G \to G$ be left & right multiplication, respectively; i.e., $L_g(h) = g \cdot h$ & $R_g(h) = h \cdot g$.

A v.f. $X \in \mathfrak{X}(G)$ is called **left-** (or **right-**) **invariant** if $dL_g V = V$ (or $dR_g V = V$) $\forall g \in G$.

The Lie algebra $\mathfrak{g}$ is formally the collection of all left-inv't v.f.'s on G, but it turns out that the map

$$\begin{aligned} \mathfrak{g} &\to T_eG \\ X &\mapsto X(e) \end{aligned}$$

is an isomorphism w/ inverse given by $V \in T_eG \mapsto X$ where $X(g) = (dL_g)_e V$.

Now, G acts on itself by conjugation:
$$\begin{aligned} G &\to \mathrm{Diff}(G) \\ g &\mapsto C_g \end{aligned} \qquad \text{where } C_g(h) = ghg^{-1}.$$

But now consider
$$\begin{array}{ccccc} (dC_g)_e & : & T_eG & \to & T_eG \\ \| & & \cong & & \cong \\ Ad_g & : & \mathfrak{g} & \to & \mathfrak{g} \end{array}$$
which is an invertible linear map since C_g is a diffeo. that maps e to e.

Now, as $g \in G$ varies we see that we get a map

$$\begin{aligned} Ad: G &\to GL(\mathfrak{g}) \\ g &\mapsto Ad_g \end{aligned}$$

called the **adjoint representation** or **adjoint action** of G on $\mathfrak{g}$.

Ex: If G is a matrix gp, $\left.\frac{d}{dt}\right|_{t=0} Ad_{\exp tX} Y = [X,Y]$ $\forall X, Y \in \mathfrak{g}$.

Now, if we define $\langle \cdot, \cdot \rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ by

$$(\xi, v) \mapsto \langle \xi, v \rangle := \xi(v),$$

then we can define $Ad_g^* \xi$ for $g \in G$ & $\xi \in \mathfrak{g}^*$ by

$$\langle Ad_g^* \xi, X \rangle = \langle \xi, Ad_{g^{-1}} X \rangle \left(= \xi(Ad_{g^{-1}} X)\right)$$

notice that choosing $\langle Ad_g^* \xi, X \rangle = \langle \xi, Ad_g X \rangle$ would not have given a gp homomorphism since then $\langle Ad_{gh}^* \xi, X \rangle = \langle \xi, Ad_{gh} X \rangle = \langle \xi, Ad_g(Ad_h X) \rangle$
$= \langle Ad_g^* \xi, Ad_h X \rangle$
$= \langle Ad_h^* Ad_g^* \xi, X \rangle$

giving the **coadjoint representation** (or **action**)
$$\begin{aligned} Ad^*: G &\to GL(\mathfrak{g}^*) \\ g &\mapsto Ad_g^* \end{aligned}$$

Notice that this exactly matches w/ what we saw w/ the $SO(3)$ action on S^2:

$$\mu(g(p))\begin{pmatrix}a\\b\\c\end{pmatrix} = \left\langle g(p), \begin{pmatrix}a\\b\\c\end{pmatrix}\right\rangle = \left\langle p, g^{-1}\begin{pmatrix}a\\b\\c\end{pmatrix}\right\rangle = \mu(p)\left(g^{-1}V_{(a,b,c)}\,g\right) = \mu(p)\left(Ad_{g^{-1}}(V_{(a,b,c)})\right)$$

for g $V \in so(3)$, so $\mu \circ g = Ad_g^* \circ \mu$.

Ex: We know $u(n)$ is the set of $n\times n$ skew-hermitian matrices ... after multiplying by i, might as well identify both $u(n)$ & $u(n)^*$ with the Hermitian matrices & both the adjoint & coadjoint actions are by conjugation.

Of course, each Hermitian matrix is conjugate by an elmt of $U(n)$ to a diagonal matrix, so we see that the orbits of the (co)adjoint action are determined by the spectrum $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$.