Math 676: Day 18

Just to recapitulate, an action of a Lie gp G on a symplectic mfd (M,ω) is Hamiltonian if $\exists$ a moment map

$$\mu: M \to \mathfrak{g}^*$$

s.t. $\mu(p)(V) = \langle \mu(p), V\rangle =: \mu^V(p)$, where μ^V is the Hamiltonian fctn for $V^\#$

& $\mu \circ g = Ad_g^* \circ \mu \quad \forall g \in G$.

Ex: As we've shown, when $G = SO(3)$ & $(M,\omega) = (S^2, \omega_{std})$, the moment map $\mu: S^2 \to \mathfrak{so}(3)^* \cong \mathbb{R}^3$

is given by $\mu(p) = p$ under the obvious identification of $\mathfrak{so}(3)^*$ w/ $\mathbb{R}^3$.

(more precisely, $\mu(p)(V_{(a,b,c)}) = p \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix}$)

Ex: Consider the diagonal $SO(3)$ action on $(S^2)^n$. For $V = V_{(a,b,c)} \in \mathfrak{so}(3)$, we know $\{\exp tV : t \in \mathbb{R}\}$ is the subgp which rotates each sphere around the axis $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$, so $V^\#(p_1, \dots, p_n) = \left(\begin{pmatrix} a \\ b \\ c \end{pmatrix} \times p_1, \dots, \begin{pmatrix} a \\ b \\ c \end{pmatrix} \times p_n \right)$

and $\iota_{V^\#}\omega = \iota_{V^\#}(\pi_1^*\omega_{std} + \dots + \pi_n^*\omega_{std}) = \omega_{std}(V_1^\#, \cdot) + \dots + \omega_{std}(V_n^\#, \cdot) = a\,dx_1 + b\,dy_1 + c\,dz_1 + \dots + a\,dx_n + b\,dy_n + c\,dz_n$

so the Hamiltonian $\mu^V(p_1, \dots, p_n) = a(x_1 + \dots + x_n) + b(y_1 + \dots + y_n) + c(z_1 + \dots + z_n) = \left\langle p_1 + \dots + p_n, \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right\rangle$,

& so we see that $\mu(p_1, \dots, p_n) = p_1 + \dots + p_n$

And now, finally, we've justified our interest in the map μ, which we saw before since the closed polygons are $\mu^{-1}(\vec{0})$.

Ex: Consider $\mathbb{R}^3$ acting on $(T^*\mathbb{R}^3, \omega) \cong (\mathbb{R}^6, \Sigma\, dx_i \wedge dy_i)$ by translating: $\Psi: \mathbb{R}^3 \to Sympl(\mathbb{R}^6, \omega)$, $\vec{v} \mapsto \Psi_{\vec{v}}$

where $\Psi_{\vec{v}}(\vec{x}, \vec{y}) = (\vec{x} + \vec{v}, \vec{y})$. Then the Lie alg. of $\mathbb{R}^3$ is just $\mathbb{R}^3$ w/ the trivial bracket, so $V = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$

$\rightsquigarrow V^\# = a\frac{\partial}{\partial x_1} + b\frac{\partial}{\partial x_2} + c\frac{\partial}{\partial x_3}$, so $\iota_{V^\#}\omega = a\,dy_1 + b\,dy_2 + c\,dy_3 = d(\langle V, \vec{y}\rangle)$

& we have a moment map $\mu: \mathbb{R}^6 \to \mathbb{R}^3$ given by $\mu(\vec{x}, \vec{y}) = \vec{y}$, since

$$\mu^V(\vec{x}, \vec{y}) = \langle \mu(\vec{x}, \vec{y}), V\rangle = \langle \vec{y}, V\rangle$$

is the Hamiltonian fctn for $V^\#$. Of course, $\vec{y}$ is the linear momentum of the system.

Ex: Consider $SO(3)$ acting on $T^*\mathbb{R}^3 \cong \mathbb{R}^6$ by rotation. We've just seen that $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 \rightsquigarrow V_{(a,b,c)} \in \mathfrak{so}(3)$

& the associated v.f. $V^\#$ on $\mathbb{R}^6$ is $V^\#(\vec{x},\vec{y}) = \left(\begin{pmatrix} a \\ b \\ c \end{pmatrix} \times \vec{x}, \begin{pmatrix} a \\ b \\ c \end{pmatrix} \times \vec{y} \right)$

Then $\mu : \mathbb{R}^6 \to \mathbb{R}^3 \cong \mathfrak{so}(3)^*$ given by $\mu(\vec{x},\vec{y}) = \vec{x} \times \vec{y}$ is a moment map (& hence the action is Hamiltonian) since

$$\left\langle \mu(\vec{x},\vec{y}), \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right\rangle = \left\langle \vec{x} \times \vec{y}, \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right\rangle$$

is the Hamiltonian function for $V^\#_{(a,b,c)}$. Of course, $\vec{x} \times \vec{y} = \mu(\vec{x},\vec{y})$ is usually called the angular momentum.

These last 2 examples show where the terminology moment(um) map comes from.

When G is connected, we can equivalently define Hamiltonian actions in terms of the existence of a comoment map

$$\mu^* : \mathfrak{g} \to C^\infty(M) \qquad \text{s.t.}$$

① $\mu^*(X) := \mu^X$ is the Hamiltonian function for $X^\#$

② μ^* is a Lie alg. homomorphism: $\mu^*[X,Y] = \{\mu^* X, \mu^* Y\}$ where $\{\,,\}$ is the Poisson bracket on $C^\infty(M)$

given by $\{f,g\} = \omega(X_f, X_g)$, where X_f & X_g are the Hamiltonian vector fields associated to f & g.

Of course, we still need to define, e.g., the coadjoint action of G on $\mathfrak{g}^*$ . . .