Calculations on SO(3) and $\mathfrak{so}(3)$

By earlier calculations, we know that the tangent space to the identity in SO (n) can be identified with the space of skew-symmetric $n \times n$ matrices. Hence, the tangent space to the identity of SO(3) consists of matrices of the form

$$\begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix},$$

which of course we can identify with an arbitrary point $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ in $\mathbb{R}^3$. Of course, the tangent space to the identity can be identified with the Lie algebra $\mathfrak{so}(3)$ (which in principle consists of left-invariant vector fields on SO(3)).

So now here's a function to turn an element of $\mathbb{R}^3$ into an element of $\mathfrak{so}(3)$:

```
so3elt[{x_, y_, z_}] := {{0, -z, y}, {z, 0, -x}, {-y, x, 0}};
```

The cross product of elements $\mathbb{R}^3$ turned into an element of of $\mathfrak{so}(3)$:

```
so3bracket[{x1_, y1_, z1_}, {x2_, y2_, z2_}] := Module[{g1, g2},
    g1 = so3elt[{x1, y1, z1}];
    g2 = so3elt[{x2, y2, z2}];
    g1.g2 - g2.g1
   ];
```

And we can see that the two are the same, and hence that ($\mathfrak{so}(3)$, [,]) and $(\mathbb{R}^3, \times)$ are isomorphic as Lie algebras:

```
so3bracket[{x1, y1, z1}, {x2, y2, z2}] // MatrixForm
```

$$\begin{pmatrix} 0 & x_2 y_1 - x_1 y_2 & x_2 z_1 - x_1 z_2 \\ -x_2 y_1 + x_1 y_2 & 0 & y_2 z_1 - y_1 z_2 \\ -x_2 z_1 + x_1 z_2 & -y_2 z_1 + y_1 z_2 & 0 \end{pmatrix}$$

```
so3elt[Cross[{x1, y1, z1}, {x2, y2, z2}]] // MatrixForm
```

$$\begin{pmatrix} 0 & x_2 y_1 - x_1 y_2 & x_2 z_1 - x_1 z_2 \\ -x_2 y_1 + x_1 y_2 & 0 & y_2 z_1 - y_1 z_2 \\ -x_2 z_1 + x_1 z_2 & -y_2 z_1 + y_1 z_2 & 0 \end{pmatrix}$$

A general 1-parameter subgroup of SO(3) is of the form $\exp(t\,V)$ for $V \in \mathfrak{so}(3)$, so has the following form:

```
subgroup = MatrixExp[t * so3elt[{a, b, c}]] // FullSimplify
```

$$\left\{\left\{\frac{a^2+(b^2+c^2)\,\mathrm{Cosh}\left[\sqrt{-a^2-b^2-c^2}\ t\right]}{a^2+b^2+c^2},\right.\right.$$
$$\frac{a\,b-a\,b\,\mathrm{Cosh}\left[\sqrt{-a^2-b^2-c^2}\ t\right]+c\,\sqrt{-a^2-b^2-c^2}\ \mathrm{Sinh}\left[\sqrt{-a^2-b^2-c^2}\ t\right]}{a^2+b^2+c^2},$$
$$\left.\frac{a\,c-a\,c\,\mathrm{Cosh}\left[\sqrt{-a^2-b^2-c^2}\ t\right]-b\,\sqrt{-a^2-b^2-c^2}\ \mathrm{Sinh}\left[\sqrt{-a^2-b^2-c^2}\ t\right]}{a^2+b^2+c^2}\right\},$$
$$\left\{\frac{a\,b-a\,b\,\mathrm{Cosh}\left[\sqrt{-a^2-b^2-c^2}\ t\right]-c\,\sqrt{-a^2-b^2-c^2}\ \mathrm{Sinh}\left[\sqrt{-a^2-b^2-c^2}\ t\right]}{a^2+b^2+c^2},\right.$$
$$\frac{b^2+(a^2+c^2)\,\mathrm{Cosh}\left[\sqrt{-a^2-b^2-c^2}\ t\right]}{a^2+b^2+c^2},$$
$$\left.\frac{b\,c-b\,c\,\mathrm{Cosh}\left[\sqrt{-a^2-b^2-c^2}\ t\right]+a\,\sqrt{-a^2-b^2-c^2}\ \mathrm{Sinh}\left[\sqrt{-a^2-b^2-c^2}\ t\right]}{a^2+b^2+c^2}\right\},$$
$$\left\{\frac{a\,c-a\,c\,\mathrm{Cosh}\left[\sqrt{-a^2-b^2-c^2}\ t\right]+b\,\sqrt{-a^2-b^2-c^2}\ \mathrm{Sinh}\left[\sqrt{-a^2-b^2-c^2}\ t\right]}{a^2+b^2+c^2},\right.$$
$$\frac{b\,c-b\,c\,\mathrm{Cosh}\left[\sqrt{-a^2-b^2-c^2}\ t\right]-a\,\sqrt{-a^2-b^2-c^2}\ \mathrm{Sinh}\left[\sqrt{-a^2-b^2-c^2}\ t\right]}{a^2+b^2+c^2},$$
$$\left.\left.\frac{c^2+(a^2+b^2)\,\mathrm{Cosh}\left[\sqrt{-a^2-b^2-c^2}\ t\right]}{a^2+b^2+c^2}\right\}\right\}$$

Now, the action on a point in S^2 is:

```
subgroup.{x, y, z}
```

$$\left\{\frac{x\left(a^2+(b^2+c^2)\,\mathrm{Cosh}\left[\sqrt{-a^2-b^2-c^2}\ t\right]\right)}{a^2+b^2+c^2}+\frac{1}{a^2+b^2+c^2}\right.$$
$$z\left(a\,c-a\,c\,\mathrm{Cosh}\left[\sqrt{-a^2-b^2-c^2}\ t\right]-b\,\sqrt{-a^2-b^2-c^2}\ \mathrm{Sinh}\left[\sqrt{-a^2-b^2-c^2}\ t\right]\right)+$$
$$\frac{1}{a^2+b^2+c^2}y\left(a\,b-a\,b\,\mathrm{Cosh}\left[\sqrt{-a^2-b^2-c^2}\ t\right]+c\,\sqrt{-a^2-b^2-c^2}\ \mathrm{Sinh}\left[\sqrt{-a^2-b^2-c^2}\ t\right]\right),$$
$$\frac{y\left(b^2+(a^2+c^2)\,\mathrm{Cosh}\left[\sqrt{-a^2-b^2-c^2}\ t\right]\right)}{a^2+b^2+c^2}+\frac{1}{a^2+b^2+c^2}$$
$$z\left(b\,c-b\,c\,\mathrm{Cosh}\left[\sqrt{-a^2-b^2-c^2}\ t\right]+a\,\sqrt{-a^2-b^2-c^2}\ \mathrm{Sinh}\left[\sqrt{-a^2-b^2-c^2}\ t\right]\right)+$$
$$\frac{1}{a^2+b^2+c^2}x\left(a\,b-a\,b\,\mathrm{Cosh}\left[\sqrt{-a^2-b^2-c^2}\ t\right]-c\,\sqrt{-a^2-b^2-c^2}\ \mathrm{Sinh}\left[\sqrt{-a^2-b^2-c^2}\ t\right]\right),$$
$$\frac{z\left(c^2+(a^2+b^2)\,\mathrm{Cosh}\left[\sqrt{-a^2-b^2-c^2}\ t\right]\right)}{a^2+b^2+c^2}+\frac{1}{a^2+b^2+c^2}$$
$$y\left(b\,c-b\,c\,\mathrm{Cosh}\left[\sqrt{-a^2-b^2-c^2}\ t\right]-a\,\sqrt{-a^2-b^2-c^2}\ \mathrm{Sinh}\left[\sqrt{-a^2-b^2-c^2}\ t\right]\right)+$$
$$\left.\frac{1}{a^2+b^2+c^2}x\left(a\,c-a\,c\,\mathrm{Cosh}\left[\sqrt{-a^2-b^2-c^2}\ t\right]+b\,\sqrt{-a^2-b^2-c^2}\ \mathrm{Sinh}\left[\sqrt{-a^2-b^2-c^2}\ t\right]\right)\right\}$$

So now we can differentiate and evaluate at $t = 0$:

```
D[%, t] /. t → 0
```

$$\left\{\frac{c\left(-a^2-b^2-c^2\right) y}{a^2+b^2+c^2}-\frac{b\left(-a^2-b^2-c^2\right) z}{a^2+b^2+c^2},\right.$$
$$\left.-\frac{c\left(-a^2-b^2-c^2\right) x}{a^2+b^2+c^2}+\frac{a\left(-a^2-b^2-c^2\right) z}{a^2+b^2+c^2},\ \frac{b\left(-a^2-b^2-c^2\right) x}{a^2+b^2+c^2}-\frac{a\left(-a^2-b^2-c^2\right) y}{a^2+b^2+c^2}\right\}$$

```
FullSimplify[%]
```

$\{-c\,y+b\,z,\ c\,x-a\,z,\ -b\,x+a\,y\}$

In other words, if $V \in \mathfrak{so}(3)$ corresponds to the point $(a, b, c) \in \mathbb{R}^3$, then the resulting circle group action on S^2 generates the vector field $V^\sharp$ on S^2, where

$$V^\sharp(x, y, z) = (b\,z - c\,y,\ c\,x - a\,z,\ a\,y - b\,x),$$

which one can easily check is equal to $(a, b, c) \times (x, y, z)$:

```
Cross[{a, b, c}, {x, y, z}]
```

$\{-c\,y+b\,z,\ c\,x-a\,z,\ -b\,x+a\,y\}$

Now, we check that the effect of rotating an element of $\mathbb{R}^3$ by some $g \in SO(3)$ is the same as conjugating the associated element of $\mathfrak{so}(3)$ by the same g. Since $g = \exp(t\,V)$ for some V, we can think of it as an element of the subgroup from before.

Rotating $(p, q, r) \in \mathbb{R}^3$ by g gives:

```
rotatefirst = subgroup.{p, q, r} // FullSimplify
```

$$\left\{\frac{1}{a^2+b^2+c^2}\left(a\,(a\,p+b\,q+c\,r)+\left(b^2\,p-a\,b\,q+c\,(c\,p-a\,r)\right)\operatorname{Cosh}\left[\sqrt{-a^2-b^2-c^2}\;t\right]+\sqrt{-a^2-b^2-c^2}\;(c\,q-b\,r)\operatorname{Sinh}\left[\sqrt{-a^2-b^2-c^2}\;t\right]\right),\right.$$
$$\frac{1}{a^2+b^2+c^2}\left(b\,(a\,p+b\,q+c\,r)+\left(-a\,b\,p+a^2\,q+c\,(c\,q-b\,r)\right)\operatorname{Cosh}\left[\sqrt{-a^2-b^2-c^2}\;t\right]+\sqrt{-a^2-b^2-c^2}\;(-c\,p+a\,r)\operatorname{Sinh}\left[\sqrt{-a^2-b^2-c^2}\;t\right]\right),$$
$$\left.\frac{1}{a^2+b^2+c^2}\left(c\,(a\,p+b\,q+c\,r)+\left(-c\,(a\,p+b\,q)+\left(a^2+b^2\right) r\right)\operatorname{Cosh}\left[\sqrt{-a^2-b^2-c^2}\;t\right]+\sqrt{-a^2-b^2-c^2}\;(b\,p-a\,q)\operatorname{Sinh}\left[\sqrt{-a^2-b^2-c^2}\;t\right]\right)\right\}$$

On the other hand, turning (p, q, r) into an element of $\mathfrak{so}(3)$ and then conjugating by g gives:

```
conjugate = FullSimplify[subgroup.so3elt[{p, q, r}].Inverse[subgroup]]
```

$$\Big\{\Big\{0,\ \frac{1}{a^2+b^2+c^2}\Big(-c\,(a\,p+b\,q+c\,r)+\big(c\,(a\,p+b\,q)-(a^2+b^2)\,r\big)\,\mathrm{Cosh}\Big[\sqrt{-a^2-b^2-c^2}\ t\Big]+\sqrt{-a^2-b^2-c^2}\ (-b\,p+a\,q)\,\mathrm{Sinh}\Big[\sqrt{-a^2-b^2-c^2}\ t\Big]\Big),$$

$$\frac{1}{a^2+b^2+c^2}\Big(b\,(a\,p+b\,q+c\,r)+\big(-a\,b\,p+a^2\,q+c\,(c\,q-b\,r)\big)\,\mathrm{Cosh}\Big[\sqrt{-a^2-b^2-c^2}\ t\Big]+\sqrt{-a^2-b^2-c^2}\ (-c\,p+a\,r)\,\mathrm{Sinh}\Big[\sqrt{-a^2-b^2-c^2}\ t\Big]\Big)\Big\},$$

$$\Big\{\frac{1}{a^2+b^2+c^2}\Big(c\,(a\,p+b\,q+c\,r)+\big(-c\,(a\,p+b\,q)+(a^2+b^2)\,r\big)\,\mathrm{Cosh}\Big[\sqrt{-a^2-b^2-c^2}\ t\Big]+\sqrt{-a^2-b^2-c^2}\ (b\,p-a\,q)\,\mathrm{Sinh}\Big[\sqrt{-a^2-b^2-c^2}\ t\Big]\Big),\ 0,$$

$$-\frac{1}{a^2+b^2+c^2}\Big(a\,(a\,p+b\,q+c\,r)+\big(b^2\,p-a\,b\,q+c\,(c\,p-a\,r)\big)\,\mathrm{Cosh}\Big[\sqrt{-a^2-b^2-c^2}\ t\Big]+\sqrt{-a^2-b^2-c^2}\ (c\,q-b\,r)\,\mathrm{Sinh}\Big[\sqrt{-a^2-b^2-c^2}\ t\Big]\Big)\Big\},$$

$$\Big\{-\frac{1}{a^2+b^2+c^2}\Big(b\,(a\,p+b\,q+c\,r)+\big(-a\,b\,p+a^2\,q+c\,(c\,q-b\,r)\big)\,\mathrm{Cosh}\Big[\sqrt{-a^2-b^2-c^2}\ t\Big]+\sqrt{-a^2-b^2-c^2}\ (-c\,p+a\,r)\,\mathrm{Sinh}\Big[\sqrt{-a^2-b^2-c^2}\ t\Big]\Big),$$

$$\frac{1}{a^2+b^2+c^2}\Big(a\,(a\,p+b\,q+c\,r)+\big(b^2\,p-a\,b\,q+c\,(c\,p-a\,r)\big)\,\mathrm{Cosh}\Big[\sqrt{-a^2-b^2-c^2}\ t\Big]+\sqrt{-a^2-b^2-c^2}\ (c\,q-b\,r)\,\mathrm{Sinh}\Big[\sqrt{-a^2-b^2-c^2}\ t\Big]\Big),\ 0\Big\}\Big\}$$

To compare these two things, we'll turn the rotated element of $\mathbb{R}^3$ into an element of $\mathfrak{so}(3)$ and then subtract:

```
so3elt[rotatefirst] - conjugate // FullSimplify
```

```
{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
```

There you go.