Math 676: Day 17

We saw last time that the $SO(3)$ action on (S^2, ω_{std}) can really be said to be Hamiltonian & that

Hamiltonian $\iff$ $V \in \mathfrak{so}(3) \mapsto V^\#$ a Hamiltonian v.f. on S^2 $\iff$ comoment map $\mu^*: \mathfrak{so}(3) \to C^\infty(S^2)$ s.t. $\mu^*(V) = \mu^V$ is Hamiltonian function for $V^\#$.

Now, let's explore a little deeper. Suppose $V_{(a,b,c)} = \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix} \in \mathfrak{so}(3)$, w/ associated circle subgroup $\exp tV_{(a,b,c)}$.

Then $V^\#_{(a,b,c)}(x,y,z) = \frac{d}{dt}\Big|_{t=0} \exp(tV_{(a,b,c)})(x,y,z) = \ldots = (bz-cy, cx-az, ay-bx) = (bz-cy)\frac{\partial}{\partial x} + (cx-az)\frac{\partial}{\partial y} + (ay-bx)\frac{\partial}{\partial z}$.

Note, first of all, that this is just the vector $\overset{\text{rotation axis}}{(a,b,c)} \times \overset{\text{point in } S^2}{(x,y,z)}$, which is obviously in $T_{(x,y,z)}S^2$ since it's $\perp$ to (x,y,z).

In other words, $V^\#(p) = V \times p$, where we identify $V \in \mathfrak{so}(3)$ as an elt. of $\mathbb{R}^3$.

Now, what is the associated Hamiltonian function μ^V?

Well, we can write the std. area form ω given by $\omega_p(u,v) = \langle p, u \times v \rangle$ as

$$\omega_{(x,y,z)} = x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy$$

Check: $(x\,dy\wedge dz + y\,dz \wedge dx + z\,dx \wedge dy)\left(u_1\frac{\partial}{\partial x} + u_2 \frac{\partial}{\partial y} + u_3 \frac{\partial}{\partial z}, v_1 \frac{\partial}{\partial x} + v_2 \frac{\partial}{\partial y} + v_3 \frac{\partial}{\partial z}\right) = x(u_2v_3 - u_3v_2) + y(u_3v_1 - u_1v_3) + z(u_1v_2 - u_2v_1) = \left\langle \begin{pmatrix} x \\ y \\ z\end{pmatrix}, u \times v \right\rangle$.

So since $(V^\#_{(a,b,c)})(x,y,z) = (bz-cy)\frac{\partial}{\partial x} + (cx-az)\frac{\partial}{\partial y} + (ay-bx)\frac{\partial}{\partial z}$, we have

$$(\iota_{V^\#}\omega)_{(x,y,z)} = (x\,dy\wedge dz + y\,dz\wedge dx + z\,dx \wedge dy)\left((bz-cy)\frac{\partial}{\partial x} + (cx-az)\frac{\partial}{\partial y} + (ay-bx)\frac{\partial}{\partial z}, \cdot\right)$$

$$= y(cy-bz)\,dz + z(bz-cy)\,dy + x(cx-az)\,dz + z(az-cx)\,dx + x(bx-ay)\,dy + y(ay-bx)\,dx$$

$$= (az^2 - cxz + ay^2 - bxy)\,dx + (bz^2 - cyz + bx^2 - axy)\,dy + (cy^2 - byz + cx^2 - axz)\,dz$$

$$= (a(y^2+z^2) - x(by+cz))\,dx + (b(x^2+z^2) - y(ax+cz))\,dy + (c(x^2+y^2) - z(ax+by))\,dz$$

$$= (a(1-x^2) - x(by+cz))\,dx + (b(1-y^2) - y(ax+cz))\,dy + (c(1-z^2) - z(ax+by))\,dz$$

$$= a\,dx - x(ax+by+cz)\,dx + b\,dy - y(ax+by+cz)\,dy + c\,dz - z(ax+by+cz)\,dz$$

$$= a\,dx + b\,dy + c\,dz - (ax+by+cz)(x\,dx + y\,dy + z\,dz).$$

But of course $x\,dx+y\,dy+z\,dz \in \Omega^1(\mathbb{R}^3)$ is not a 1-form on S^2: it corresponds to the radial v.f. which is $\perp$ to S^2,

so in fact
$$(\iota_{V^\#}\omega)_{(x,y,z)} = a\,dx + b\,dy + c\,dz.$$

So then what's the Hamiltonian function μ^V s.t. $d\mu^V = (\iota_{V^\#}\omega)_{(x,y,z)}$?

Clearly $\mu^V(x,y,z) = ax+by+cz = \langle (a,b,c)^T, (x,y,z)^T \rangle$ or, by abuse of notation, $\mu^V(p) = \langle p, V\rangle$.

Notice, moreover, that for any $g \in SO(3)$, $\mu^{V_{(a,b,c)}}(g(p)) = \langle g(p), (a,b,c)^T\rangle = \langle p, g^{-1}(a,b,c)^T\rangle$

Now, the $\mathfrak{so}(3)$ elt. corresponding to $g^{-1}(a,b,c)^T$ is the same as $g^{-1}V_{(a,b,c)}\,g \in \mathfrak{so}(3)$, so there's equivariance b/w the $SO(3)$ action on S^2 & the $SO(3)$ action on $\mathfrak{so}(3)$.

Now, of course in general there's no way to take a dot product b/w $p \in (M,\omega)$ & $V \in \mathfrak{g}$.

So the right thing to do is really to think of the $SO(3)$ action as defining a moment map $\mu: S^2 \to \mathfrak{so}(3)^*$

where $\mu(p)(V) = \langle p, V\rangle = \mu^V(p)$ is the Hamiltonian fctn for $V^\#$ & so that μ is equivariant w.r.t.

the $SO(3)$ action on S^2 & the coadjoint $SO(3)$ action on $\mathfrak{so}(3)^*$.

Df: Suppose (M,ω) is symplectic, G is a Lie group w/ Lie algebra $\mathfrak{g}$, whose dual v.s. is $\mathfrak{g}^*$ & $\psi: G \to \mathrm{Sympl}(M,\omega)$ is a symplectic action. The action ψ is Hamiltonian if $\exists\, \mu: M \to \mathfrak{g}^*$ s.t.

① $\forall\, X \in \mathfrak{g}$, let $\mu^X: M \to \mathbb{R}$ given by $\mu^X(p) := \langle \mu(p), X\rangle$ be the component of μ along X

and let $X^\# \in \mathfrak{X}(M)$ be the v.f. generated by the one-parameter subgroup $\{\exp tX : t \in \mathbb{R}\} \subseteq G$.

Then $d\mu^X = \iota_{X^\#}\omega$. In other words, μ^X is the Hamiltonian fctn for $X^\#$.

② μ is equivariant w.r.t. the action ψ of G on M & the coadjoint action Ad^* of G on $\mathfrak{g}^*$:

$$\mu \circ \psi_g = \mathrm{Ad}_g^* \circ \mu \quad \forall\, g \in G.$$

Given such a Hamiltonian action, the quadruple (M,ω,G,μ) is called a Hamiltonian G-space & μ is a moment map.

When G is connected, we can equivalently define Hamiltonian actions in terms of the existence of a comoment map

$$\mu^*: \mathfrak{g} \to C^\infty(M) \quad \text{s.t.}$$

① $\mu^*(X) := \mu^X$ is the Hamiltonian function for $X^\#$

② μ^* is a Lie alg. homomorphism: $\mu^*[X,Y] = \{\mu^*X, \mu^*Y\}$ where $\{,\}$ is the Poisson bracket on $C^\infty(M)$

given by $\{f,g\} = \omega(X_f, X_g)$, where X_f & X_g are the Hamiltonian vector fields associated to f & g.

Of course, we need to define some of these symbols & terms...