Math 676: Day 16

Recall: we've defined an action $\varphi: \underbrace{S^1 \times \dots \times S^1}_{n} \to \mathrm{Sympl}(M,\omega)$ to be Hamiltonian if each $\varphi^i : S^1_i \to \mathrm{Sympl}(M,\omega)$

is Hamiltonian, meaning $\exists\, H_i : M \to \mathbb{R}$ s.t. $dH_i = \iota_{V_i}\omega$, where $V_i := \frac{d\varphi^i_t}{dt}\Big|_{t=0}$ is the v.f. associated to the S^1-action φ^i.

Of course then we can package the H_i's into a single function $H: M \to \mathbb{R}^n$ s.t. $H(p) = (H_1(p), \dots, H_n(p))$.

We will shortly rename H as μ, for moment map.

Now, what should Hamiltonian mean for a nonabelian Lie group action?

Let's work this out in the case of $G = SO(3)$, which is the most important noncommutative Lie group in the random walks story.

Also, we'll let $(M,\omega) = (S^2, \omega_{std})$; in the random walks story, the symplectic mfld will just be the product $\underbrace{S^2 \times \dots \times S^2}_{n}$,

the symplectic form will be the sum of ω_{std} on each factor, & the $SO(3)$ action will be the diagonal action.

Hence, understanding $SO(3) \circlearrowright (S^2, \omega_{std})$ gets us 90% of the way there.

Now, rotations of S^2 are obviously area-preserving & hence symplectic, but what should the Hamiltonian condition be?

Obviously, a Hamiltonian $SO(3)$ action should restrict to a Hamiltonian action in the previously-defined sense on each abelian subgroup.

Now, the maximal abelian subgroups of $SO(3)$ are all conjugate to $SO(2)$... geometrically speaking, they correspond to rotation

around some axis.

We saw before that rotation around the z-axis was Hamiltonian, w/ associated Hamiltonian function $H: S^2 \to \mathbb{R}$ given by $H(x,y,z) = z$.

Now, if we're rotating around some axis determined by the unit vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$, what is the corresponding conserved quantity?

Well, it's the dot product w/ $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$, so the associated Hamiltonian function should be $H(x,y,z) = \left\langle \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right\rangle$.

So clearly we see that each circle subgroup action **is** Hamiltonian... but let's look at this more abstractly...

Any 1-parameter subgroup of $SO(3)$ is determined by its derivative at the identity: they are of the form $\exp tV$ for

$$V \in T_I SO(3) \cong \mathfrak{so}(3).$$

We saw before (Day 4 notes) that $T_I SO(n)$ is the set of skew-symmetric $n \times n$ matrices, so

$$\mathfrak{so}(3) = \left\{ \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix} ; x,y,z \in \mathbb{R} \right\}.$$

This particular choice of coordinates is made so that: ① $[V_{(x,y,z)}, V_{(a,b,c)}] = V_{(x,y,z) \times (a,b,c)}$; i.e.
$(\mathfrak{so}(3), [\,,\,]) \cong (\mathbb{R}^3, \times)$ as Lie algebras

② $\exp tV_{(x,y,z)}$ corresponds to rotation around (x,y,z); if (x,y,z) is a unit vector, then $\exp t_0 V_{(x,y,z)}$ is rotation around (x,y,z) by the angle t_0.

Now, given $V \in \mathfrak{so}(3)$, we get a vector field $V^\# \in \mathfrak{X}(S^2)$ s.t. $V^\#(p) = \frac{d}{dt}\Big|_{t=0} (\exp tV)(p)$

So then a reasonable notion of Hamiltonian would be that $V^\#$ is always a Hamiltonian v.f. (i.e. $\iota_{V^\#}\omega$ is exact) $\forall\, V \in \mathfrak{so}(3)$

In fact, can go a little further: if $V^\#$ is always Hamiltonian, then there's always an associated Hamiltonian function which I'll call μ^V & so we have a comoment map $\mu^*: \mathfrak{so}(3) \to C^\infty(S^2)$ s.t. $\mu^*(V) := \mu^V$ is the Hamiltonian function for $V^\#$.