Math 476: Day 15

Recall from last time that, given a mfld M, there's a correspondence

$$\text{v.f. } X \longleftrightarrow \text{flow } \varphi_t \longleftrightarrow \mathbb{R}\text{-action } \varphi: \mathbb{R} \to \mathrm{Diff}(M), \quad t \mapsto \varphi_t$$

If we have a symplectic $\mathbb{R}$-action $\psi: \mathbb{R} \to \mathrm{Sympl}(M)$, then the corresponding complete v.f. X is a symplectic vector field, meaning that $\mathcal{L}_X \omega = 0$ or, equivalently (by Cartan's Magic Formula) $\iota_X \omega$ is closed.

Ex: Consider the std. square torus $T^2 = S^1 \times S^1$. This is an orientable surface, so can be made symplectic by choosing an area form... say $\omega = d\theta_1 \wedge d\theta_2$

Now, consider the $\mathbb{R}$-action $\psi: \mathbb{R} \to \mathrm{Diff}(M)$ given by $\psi_t(\theta_1, \theta_2) = (\theta_1 + t, \theta_2)$.

Then the corresponding v.f. X is given by $X_{(\theta_1,\theta_2)} = \left.\frac{d\psi_t(\theta_1,\theta_2)}{dt}\right|_{t=0} = \left.\frac{d}{dt}\right|_{t=0}(\theta_1 + t, \theta_2) = \frac{\partial}{\partial \theta_1}$.

Of course, $\mathcal{L}_X \omega = d\iota_X\omega + \iota_X \cancel{d\omega}^{\,0} = d\iota_X\omega = d\left[(d\theta_1 \wedge d\theta_2)\left(\frac{\partial}{\partial\theta_1}, \cdot\right)\right] = d(d\theta_2) = 0$, so $\frac{\partial}{\partial\theta_1}$ is a symplectic v.f. Likewise, $\frac{\partial}{\partial\theta_2}$ is symplectic.

Before, we saw that a smooth fctn $H: M \to \mathbb{R}$ has an associated Hamiltonian v.f. X_H s.t. $\iota_{X_H}\omega = dH$.

Of course, $\iota_{X_H}\omega = dH \Rightarrow d\iota_{X_H}\omega = ddH = 0$, so X_H is symplectic.

In genl, a symplectic v.f. X is Hamiltonian if $\iota_X\omega$ is exact, meaning $\exists$ smooth $H: M \to \mathbb{R}$ s.t. $\iota_X \omega = dH$; then H is called the Hamiltonian fctn of X.

(Recall $JX_H = \nabla H$ on $(\mathbb{R}^{2n}, \omega_{std})$, so symplectic is the analogue of curl-free & Hamiltonian is the analogue of a being a potential field, where the Hamiltonian fctn is like the potential fctn)

Ex: Again, on $(S^2, d\theta \wedge dz)$, we have the $\mathbb{R}$-action given by rotating around the z-axis, which has associated v.f. $\frac{\partial}{\partial\theta}$, which we already know is Hamiltonian w/ Hamiltonian fctn $H(\theta, z) = z$. Of course, this $\mathbb{R}$-action is 2π-periodic, so descends to an S^1 action, which is symplectic & in fact Hamiltonian.

Def: A symplectic S^1 action ψ on (M,ω) is Hamiltonian if the v.f. generated by ψ is a Hamiltonian v.f.

Equivalently, an action ψ of S^1 or $\mathbb{R}$ on (M,ω) is Hamiltonian if $\exists$ smooth $H: M \to \mathbb{R}$ w/ $dH = \iota_X \omega$, where X is the v.f. generated by ψ.

Of course, an action of $G = \begin{cases} \underbrace{S^1 \times \ldots \times S^1}_{n} \\ \underbrace{\mathbb{R} \times \ldots \times \mathbb{R}}_{n} \end{cases}$ is Hamiltonian if the action of each factor is Hamiltonian.

Ex: Notice that the previously-defined action of S^1 on $T^2 = S^1 \times S^1$ given by $\psi_t(\theta_1, \theta_2) = (\theta_1 + t, \theta_2)$ is not Hamiltonian since $\iota_X \omega = d\theta_1$, but θ_1 is not a smooth function on T^2 (& indeed $d\theta_1$ represents a nontrivial cohomology class of the torus).