Math 476: Day 14

Last time I gave (a sense of) the justification that the trajectories of the Hamiltonian vector field X_H associated to a smooth map $H:(M,\omega)\to\mathbb{R}$ preserve the value of H.

Key Example: Consider $M=S^2$ w/ the standard symplectic structure $\omega=d\theta\wedge dz$, & consider the height function $H(\theta,z)=z$.

Then $\iota_{X_H}\omega=\iota_{X_H}d\theta\wedge dz=dH=dz$, so $X_H=\frac{\partial}{\partial\theta}$ & $\varphi_t(\theta,z)=(\theta+t,z)$ is rotation around the z-axis,

which obviously preserves the height.

Okay, now notice in this case that the flow of X_H is periodic: $\varphi_t\equiv\varphi_{t+2\pi}$.

In fact, the trajectories (integral curves) of X_H are also exactly the orbits of the S^1 action which rotates the sphere around the z-axis.

Said yet another way, the circle action gives a (1-parameter) family of symmetries of the sphere, & the function H gives a corresponding conserved quantity.

Noether's Principle: Every symmetry of a mechanical system has a corresponding conserved quantity.

Since (by Darboux's Thm) every symplectic mfd looks locally like $(\mathbb{R}^{2n},\omega_{std})$ & hence like the phase space of a mechanical system, Noether's Principle should hold, in some sense, on all symplectic mfds.

We will see that it does, but first notice the consequence: given a mfd w/ a symmetry, the associated conserved quantity gives a coordinate on the mfd ... & if you have lots of symmetries, you get lots of coordinates.

The key example here is a toric symplectic mfd: (M^{2n},ω) w/ an action $(S^1)^n\curvearrowright(M,\omega)$ which preserves the symplectic structure.

Then we should have n conserved quantities... even better, on the orbits you can just push forward the angle coords. on the torus, giving $n+n=2n$ coords on (M,ω), which is what you want!

Okay, so the moral of the time is that group actions are good. Also, recall that our polygon space is

$\mu^{-1}(\vec{0})/SO(3)$, where $SO(3)$ acts diagonally, so we need to understand gp actions on symplectic manifolds anyway.

If M is a mfld & $X \in \mathfrak{X}(M)$ is complete (e.g., if M is cpct), then we have the associated isotopy $\varphi: M \times \mathbb{R} \to M$

which solves the initial value problem

$$\begin{cases} \varphi_0 \equiv 1_M \\ \frac{d\varphi_t}{dt} = X \circ \varphi_t \end{cases}$$

Lemma: $\varphi_t \circ \varphi_s = \varphi_{t+s} \quad \forall\, s,t \in \mathbb{R}$.

Cor: $\varphi_t^{-1} = \varphi_{-t}$

Cor: $\mathbb{R} \to \mathrm{Diff}(M)$ given by $t \mapsto \varphi_t$ is a group homomorphism.

The image $\{\varphi_t : t \in \mathbb{R}\}$ is called a one-parameter group of diffeomorphisms.

Now, $\mathbb{R}$ is a Lie group. In general, a Lie group is a mfld G which is also a group so that

$$\begin{aligned} G \times G &\to G \\ (a,b) &\mapsto ab \end{aligned} \qquad \begin{aligned} G &\to G \\ a &\mapsto a^{-1} \end{aligned}$$

are smooth maps.

Moreover, the map $\mathbb{R} \to \mathrm{Diff}(M)$ defined above is an example of a Lie group action. In gen'l, an action of the Lie group G on the mfld M is a group homomorphism $\psi: G \to \mathrm{Diff}(M)$, $g \mapsto \psi_g$ (here this is implicitly a left action)

There's an associated evaluation map

$$\begin{aligned} ev_\psi : M \times G &\to G \\ (p,g) &\mapsto \psi_g(p) \end{aligned}$$

The action is smooth if ev_ψ is a smooth map.

Obviously, in the case of a cplete v.f. X on M, the action is smooth; conversely, all smooth $\mathbb{R}$-actions on M are defined

by an associated cplete v.f.: $\psi \longmapsto X_p = \left.\frac{\partial \psi_t(p)}{\partial t}\right|_{t=0}$

Now, a smooth action ψ of G on a symplectic mfld (M,ω) is symplectic if the image of $\psi: G \to \mathrm{Diff}(M)$ is contained

in $\mathrm{Sympl}(M)$, the group of symplectomorphisms of (M,ω).