Math 676: Day 13

Flows, vector fields, & Lie derivatives

Recall in our discussion of Hamiltonian mechanics that we realized the evolution of a physical system as a trajectory of a Hamiltonian v.f., & that the flow of the vector field both conserved the energy & acted by symplectomorphisms on phase space.

We can now give some better justification for that:

Let M be a mfld & $\varphi: M\times\mathbb{R}\to M$ a map, where $\varphi_t(p) := \varphi(p,t)$. The map φ is an isotopy if each $\varphi_t: M\to M$ is a diffeo. & $\varphi_0 = \mathbb{1}_M$.

Given an isotopy φ, we get a time-dependent vector field $V_t\in\mathfrak{X}(M)$ $\forall t$ s.t.

$$V_t(p) = \frac{d}{ds}\Big|_{s=t}\varphi_s(\varphi_t^{-1}(p)) \qquad \text{or, equivalently,} \qquad \frac{d\varphi_t}{dt} = V_t\circ\varphi_t \quad (*)$$

OTOH, if M is cpct & V_t is a time-dependent v.f., $\exists$ isotopy φ satisfying the same differential equation, so we have a 1-1 correspondence b/w time-dependent v.f.'s & isotopies on cpct manifolds.

If $V_t\equiv V$ is independent of t, the isotopy φ_t is called the flow of V. Of course then $(*)$ becomes

$$\frac{\partial\varphi_t}{\partial t} = V_{\varphi_t(p)},$$

which you hopefully recognize as the local flow of V from the Day 4 notes.

Now, some other terminology: φ_t is equal to $\exp(tV)$, where $\exp$ denotes the usual exponential map.

Equivalently, can define $\exp(tV)$ by:

$$\exp(tV)\Big|_{t=0} = \mathbb{1}_M \quad \& \quad \frac{d}{dt}\exp(tV)(p) = V((\exp(tV))(p)) \quad \} \text{ just substituting } \exp(tV) \text{ for } \varphi_t \text{ in } (*)$$

Def: For $V\in\mathfrak{X}(M)$, the Lie derivative on k-forms on M is the operator $\mathcal{L}_V: \Omega^k(M)\to\Omega^k(M)$ given by

$$\mathcal{L}_V\eta := \frac{d}{dt}\Big|_{t=0}(\exp tV)^*\eta = \frac{d}{dt}\Big|_{t=0}\varphi_t^*\eta$$

Compare to the expression of the Lie bracket as a differential operator on v.f.'s:

$$[X,Y]_p = \lim_{t\to 0}\frac{Y_{\alpha_t(p)} - d\alpha_t(Y_p)}{t} = \frac{d}{dt}\Big|_{t=0} d\varphi_{-t}Y$$

In general, if V_t is time-dependent, still get $\mathcal{L}_{V_t}\eta := \frac{d}{dt}\Big|_{t=0}(\varphi_t)^*\eta$.

Cartan's Magic Formula: $$\mathcal{L}_V\eta = \iota_V d\eta + d\iota_V\eta$$

Pf: Exercise

Also, $\frac{d}{dt}\varphi_t^*\omega = \varphi_t^*\mathcal{L}_{V_t}\omega$ for any time-dependent v.f. V_t.

This now justifies that statement that flows of Hamiltonian v.f.'s preserve the symplectic structure: if (M,ω) symplectic, $H: M\to\mathbb{R}$ smooth & $\exists\, X_H \in \mathfrak{X}(M)$ s.t. $\iota_{X_H}\omega = dH$ & φ_t is the flow associated to X_H, then

$$\frac{d}{dt}\varphi_t^*\omega = \varphi_t^*\mathcal{L}_{X_H}\omega = \varphi_t^*(\iota_{X_H}d\omega + d\iota_{X_H}\omega) = \varphi_t^*(0 + d(dH)) = 0,$$

so $\varphi_t^*\omega = \varphi_0^*\omega = \mathbb{1}_M^*\omega = \omega$.

Likewise,

$$\frac{d(H\circ\varphi_t)(p)}{dt} = \frac{d}{dt}(\varphi_t^* H)(p) = (\varphi_t^*\mathcal{L}_{X_H}H)(p) = \varphi_t^*(\iota_{X_H}dH + d\cancel{\iota_{X_H}H}^{\,0})(p) = \varphi_t^*(\iota_{X_H}\iota_{X_H}\omega)(p)$$
$$= \varphi_t^*(\omega(X_H,X_H))(p) = 0,$$

so the function H (called the Hamiltonian or the energy) is constant along trajectories of X_H.

This is cool. If you have a smooth function on a (compact) symplectic manifold (M,ω), then this gives you an explicit way to travel along the level set of the function: just flow along X_H

(Recall that in $(\mathbb{R}^{2n}, \omega_{std})$, $JX_H = \nabla H$, so this is in some sense not surprising, since at least in this setting X_H is orthogonal to the gradient of H.

Of course this still works even when there's no Riemannian metric & hence no sense of "orthogonal," which is more interesting.)