Math 676: Day 12

Recall that, given a diffeo. $f: M_1 \to M_2$, we get a diffeo. $f_\#: T^*M_1 \to T^*M_2$ given by $f_\#(p_1,\xi_1) = (f(p_1), ((df_p)^*)^{-1}\xi_1)$.

Prop: In fact, $(f_\#)^*\alpha_2 = \alpha_1$.

Proof: Again, we prove this pointwise, so the goal is to show that $((df_\#)_{(p_1,\xi_1)})^*\alpha_{2\,(p_2,\xi_2)} = (\alpha_1)_{(p_1,\xi_1)}$ for any $(p_1,\xi_1) \in T^*M_1$.

$$
\begin{aligned}
((df_\#)_{(p_1,\xi_1)})^*\alpha_{2\,(p_2,\xi_2)} &\overset{\text{def of }\alpha_2}{=} ((df_\#)_{(p_1,\xi_1)})^*(d\pi_2)^*_{(p_2,\xi_2)}\xi_2 = (d\pi_2\, df_\#)^*\xi_2 \overset{\text{chain rule}}{=} (d(\pi_2\circ f_\#)_{(p_1,\xi_1)})^*\xi_2 \\
&\overset{\text{commutativity of diagram}}{=} (d(f\circ\pi_1)_{(p_1,\xi_1)})^*\xi_2 \\
&\overset{\text{chain rule}}{=} (df_{p_1}(d\pi_1)_{(p_1,\xi_1)})^*\xi_2 \\
&= ((d\pi_1)_{(p_1,\xi_1)})^*(df_p)^*\xi_2 \\
&\overset{\text{def. of }f_\#}{=} ((d\pi_1)_{(p_1,\xi_1)})^*\xi_1 \\
&\overset{\text{def. of }\alpha_1}{=} (\alpha_1)_{(p_1,\xi_1)}
\end{aligned}
$$

$$
\begin{tikzcd}
T^*M_1 \arrow[r, "f_\#"] \arrow[d, "\pi_1"'] & T^*M_2 \arrow[d, "\pi_2"] \\
M_1 \arrow[r, "f"] & M_2
\end{tikzcd}
$$

∎

Corollary: $f_\#: T^*M_1 \to T^*M_2$ is a symplectomorphism.

Pf: $(f_\#)^*\omega_2 = (f_\#)^*(-d\alpha_2) = -d\,(f_\#)^*\alpha_2 = -d\alpha_1 = \omega_1$. ∎

Ex: Let $M = S^1$. Then $T^*S^1 \cong S^1\times\mathbb{R}$ is an infinite cylinder. By definition, $\alpha = \xi\, d\theta$ & $\omega = -d\alpha = -d\xi\wedge d\theta = d\theta\wedge d\xi$ is the standard area form.

But then this means that, for *any* diffeomorphism $f: S^1\to S^1$ of the circle, the induced map $f_\#: (S^1\times\mathbb{R}) \to (S^1\times\mathbb{R})$ is **area-preserving**.

This is slightly surprising, since self-diffeomorphisms of the circle definitely do *not* need to preserve arclength.

Of course, if you look at the definition of $f_\#: (p_1,\xi_1) \mapsto (p_2, ((df_{p_1})^*)^{-1}\xi_1)$ you see that the cotangent direction is contracted when the circle direction is expanded & vice versa, so it works out.

Note: $f \mapsto f_\#$ defines an injective group homomorphism $\mathrm{Diff}(M) \to \mathrm{Symp}(T^*M,\omega)$.

This is not surjective, as you can see by just shifting the cylinder in the prev. example up.

Def: Suppose (M,ω) is symplectic. A submanifold Y is a **Lagrangian submanifold** if, at each $p \in Y$, $\omega_p|_{T_pY} = 0$

and $\dim Y = \frac{1}{2}\dim M$.

Now, if M^n is a mfld w/ local coords. $(x_1, \dots, x_n)$ on $U \subseteq M$ & associated cotgt coords. $x_1, \dots, x_n, \xi_1, \dots, \xi_n$ on T^*U, then

the tautological form in coords is $\alpha = \sum \xi_i dx_i$

& the canonical symplectic form is $\omega = -d\alpha = \sum dx_i \wedge d\xi_i$

The zero section $\sigma_0 : M \to T^*M$ is given by $\sigma_0(p) = (p, 0)$. Let $M_0 = \sigma_0(M)$. Obviously α vanishes on

$M_0 \cap T^*U$, so ω does as well & hence M_0 is Lagrangian in T^*M.

This is the sense in which every mfld is Lagrangian.

Cute fact: Recall that a 1-form η on M is a smooth section of the cotangent bundle. If we define $M_\eta := \eta(M) = \{(p, \eta_p)\}$

then M_η is Lagrangian $\Leftrightarrow d\eta = 0$.

Weinstein Tubular Nbd thm Let (M,ω) be symplectic, X a cpct Lagrangian submfld, ω_0 the canonical symplectic structure

on T^*X, $\sigma_0 : X \to T^*X$ the (Lagrangian) zero section, & $i : X \hookrightarrow M$ the (Lagrangian) inclusion.

$\exists$ nbhds U_0 of X in T^*X, U of X in M, & a diffeo. $\varphi : U_0 \to U$ s.t.

$$\begin{tikzcd} U_0 \arrow[rr, "\varphi"] & & U \\ & X \arrow[ul, "i_0"] \arrow[ur, "i"'] \arrow[u, phantom, "\circlearrowleft"] & \end{tikzcd} \qquad \& \quad \varphi^* \omega = \omega_0$$
