Math 676: Day 11

Mantra: All (co)tangent bundles are symplectic & all manifolds are Lagrangians

If M^n is a mfd w/ local coords $(x_1,\dots,x_n)$ near a point $p\in M$ (meaning formally that (U,φ) is a coord chart containing p & $x_i:\varphi(U)\to\mathbb{R}$ is given by $x_i(\varphi(a_1,\dots,a_n))=a_i$), then any $\xi\in T_q^*M$ can be written as

$$\xi=\sum_{i=1}^n \xi_i\,dx_i$$

for $q\in\varphi(U)$. But then we get a map $T^*\varphi(U)\to\mathbb{R}^{2n}$

$$(q,\xi)\longmapsto(x_1,\dots,x_n,\xi_1,\dots,\xi_n)$$

then $(x_1,\dots,x_n,\xi_1,\dots,\xi_n)$ give local coords on $T^*\varphi(U)$, & this makes T^*M into a manifold.

Even better, in local coords. we can define the 2-form

$$\omega=\sum_{i=1}^n dx_i\wedge d\xi_i$$

which is equal to $-d\alpha$ where $\alpha=\sum\xi_i dx_i$, & hence is closed. Moreover, it's obviously nondegenerate since it corresponds in each coord. patch to the standard symplectic form on $\mathbb{R}^{2n}$.

The form ω is called the canonical symplectic form on T^*M & α is the tautological form or Liouville 1-form.

Of course, one has to check they're independent of the particular choice of local coords., but this is true. Rather than checking, though, we define α & ω in a coordinate-free way.

Let $\begin{array}{c}T^*M\\ \pi\downarrow\\ M\end{array}\quad\begin{array}{c}(p,\xi)\\ \downarrow\\ p\end{array}$ be the natural projection. We can define α pointwise by

$$\alpha_{(p,\xi)}=(d\pi_{(p,\xi)})^*\xi=\xi\circ d\pi_{(p,\xi)}$$

Here we have

$$T_{(p,\xi)}(T^*M)\xrightarrow{d\pi_{(p,\xi)}}T_pM$$

$$T^*_{(p,\xi)}(T^*M)\xleftarrow{(d\pi_{(p,\xi)})^*}T_p^*M$$

In other words, $\alpha_{(p,\xi)}(v)=\xi((d\pi_{(p,\xi)})v)$ for any $v\in T_{(p,\xi)}M$.

Then, of course, $\omega=-d\alpha$.

Prop: α is characterized by: $\forall \mu \in \Omega^1(M)$, $\mu^*\alpha = \mu$.

What does this mean? Remember that 1-forms on M are smooth sections of $\bigcup_{p\in M} \Lambda^1(T_p^*M) = T^*M$, so

$$\begin{tikzcd} T^*M \arrow[d, "\pi"'] \\ M \arrow[u, bend right, "\mu"'] \end{tikzcd}$$

But then, μ is a smooth map, so $\mu^*: \Omega^1(T^*M) \to \Omega^1(M)$

& so $\mu^*(\alpha)$ is a 1-form on M.

Proof: We can just prove this pointwise, so let $p \in M$; the goal is to show $(\mu^*(\alpha))_p(v) = \mu_p(v)$ for any $v \in T_pM$.

But now
$$(\mu^*(\alpha))_p(v) = \alpha_{\mu(p)}(d\mu_p v) = \alpha_{(p,\mu_p)}(d\mu_p v) = \mu_p(d\pi_{(p,\mu_p)} d\mu_p v)$$
$$= \mu_p(d(\pi\circ\mu)_p v)$$
$$= \mu_p(d(\mathbb{1}_M)_p v)$$
$$= \mu_p(v).$$
∎

Now, the nifty thing is that, w/ this machinery, any diffeomorphism of manifolds turns into a canonical symplectomorphism of their cotangent bundles as follows:

If M_1, M_2 are n-manifolds w/ tautological 1-forms α_1, α_2 on T^*M_1, T^*M_2 & $f: M_1 \to M_2$ is a diffeo., then the natural lift

$$f_\#: T^*M_1 \to T^*M_2$$
$$(p_1, \xi_1) \mapsto (p_2, \xi_2)$$

where $p_2 = f(p_1)$ & $\xi_1 = (df_{p_1})^*\xi_2$ (or, equivalently, $\xi_2 = ((df_{p_1})^*)^{-1}\xi_1$)

One can check this is really a diffeo.; the only annoying bit is convincing yourself that $f_\#$ & $f_\#^{-1}$ are smooth.