Math 676: Day 10

Unlike, say, Riemannian manifolds, symplectic manifolds have no local structure (or, maybe: there's no such thing as symplectic curvature)

More precisely:

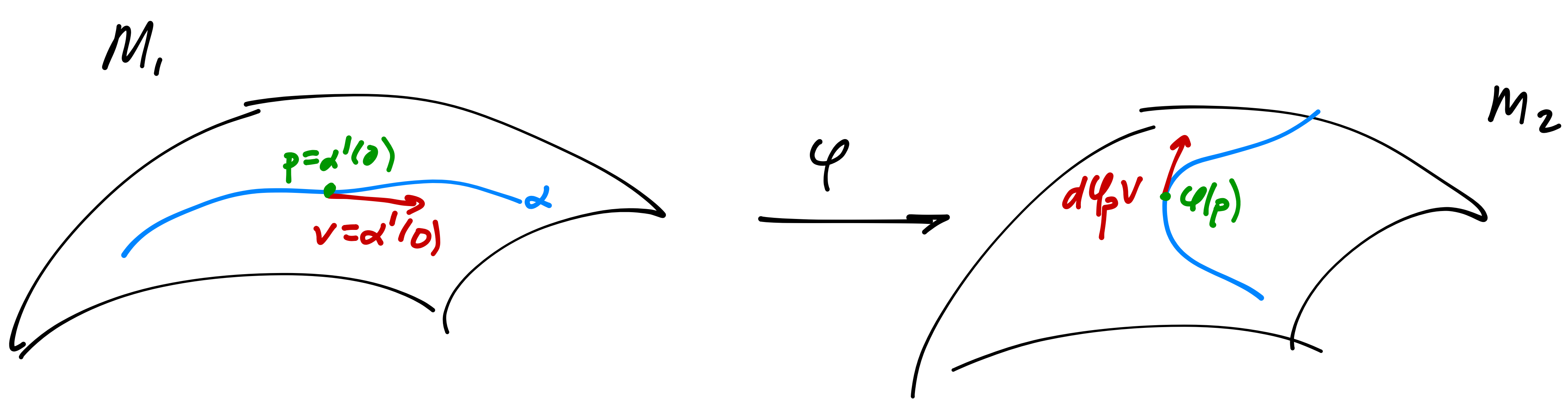
Def: Let $(M_1,\omega_1), (M_2,\omega_2)$ be $2n$-dim'l symplectic mfds. A **symplectomorphism** $\varphi:(M_1,\omega_1)\to(M_2,\omega_2)$ is a diffeomorphism $\varphi: M_1\to M_2$ s.t. $\varphi^*\omega_2=\omega_1$.

Okay, what is the **pullback** φ^*, which we haven't talked about yet?

Well, recall first the **differential** $d\varphi$ (sometimes thought of as the **pushforward**, in which case it usually gets denoted as φ_*)

which, at each $p\in M_1$, induces a map $d\varphi_p: T_pM_1\to T_{\varphi(p)}M_2$ given by

$$d\varphi_p(\alpha'(0)) = \frac{d}{dt}\Big|_{t=0}(\varphi\circ\alpha)$$



Now, we can define the **pullback** φ^* as, for each $k=0,\dots,2n$, a map $\varphi^*:\Omega^k(M_2)\to\Omega^k(M_1)$ given by

$$\varphi_q^*(\eta)(v_1,\dots,v_k) := \eta(d\varphi_q v_1,\dots,d\varphi_q v_k)$$

for each $\eta\in\Omega^k(M_2)$ & at each $q\in M_2$.

Now:

Darboux Thm: Let (M^{2n},ω) be symplectic & let $p\in M$. $\exists$ coord. chart (U,φ) centered at p w/ local coords. $x_1,y_1,\dots,x_n,y_n$ on U s.t. $\omega|_U=\sum_{i=1}^n dx_i\wedge dy_i$.

The chart that achieves this is called a **Darboux chart**.

Now, symplectic geometry grew out of Hamiltonian mechanics. Although we probably won't do much w/ that directly, it's worthwhile to learn a bit about the classical setting.

In that context, think of a collection of n particles in $\mathbb{R}^3$ having position coords. $\vec{q}_1, \dots, \vec{q}_n$ where $\vec{q}_i = (q_{i1}, q_{i2}, q_{i3})$.

So the configuration space of the system is $(\mathbb{R}^3)^n = \mathbb{R}^{3n}$ w/ coords $q_{11}, q_{12}, q_{13}, q_{21}, \dots, q_{n2}, q_{n3}$.

Moreover, assume the particles move under a potential V along curves $\vec{q}_i(t)$ s.t. $m \frac{d^2 \vec{q}_i}{dt^2} = -\nabla V(\vec{q}_i)$.

Now, these particles also have momenta $\vec{p}_i = m_i \frac{d\vec{q}_i}{dt}$ and we can package positions & momenta together & see these as coordinates of phase space $T^*\mathbb{R}^{3n} \cong \mathbb{R}^{3n} \times \mathbb{R}^{3n} \cong \mathbb{R}^{6n}$.

We also have the energy function $H(p,q) = \sum_{i=1}^{n} \left[\frac{1}{2m_i} |\vec{p}_i|^2 + V(\vec{q}_i) \right]$.

Now, what does Newton's second law say?

Of course, it says $F_i = m_i \vec{a}_i = m_i \frac{d^2}{dt^2} \vec{q}_i = \frac{d}{dt} \vec{p}_i$ or, more precisely,

$$\begin{cases} \dfrac{dq_{ij}}{dt} = \dfrac{1}{m} p_{ij} \\[2ex] \dfrac{dp_{ij}}{dt} = m_i \dfrac{d^2 q_{ij}}{dt^2} \end{cases} \qquad \text{for } i = 1, \dots, n, \quad j = 1, 2, 3$$

Okay, but now notice that $\frac{\partial H}{\partial p_{ij}} = \frac{p_{ij}}{m} = \frac{dq_{ij}}{dt}$ & $\frac{\partial H}{\partial q_{ij}} = \frac{\partial V}{\partial q_{ij}} = -m_i \frac{d^2 q_{ij}}{dt^2}$, so the above reduces to

$$\begin{cases} \dfrac{dq_{ij}}{dt} = \dfrac{\partial H}{\partial p_{ij}} \\[2ex] \dfrac{dp_{ij}}{dt} = -\dfrac{\partial H}{\partial q_{ij}} \end{cases}$$

the Hamilton equations.

Now, where's the symplectic geometry?

If we let $M = T^*\mathbb{R}^3 \cong \mathbb{R}^{6n}$ & define $\omega = \sum_{i=1}^{n} \sum_{j=1}^{3} dq_{ij} \wedge dp_{ij}$, then ω is symplectic (it's just the standard Darboux form)

Since ω is nondegenerate, there is a unique vector field X_H on M (called the Hamiltonian v.f.)

s.t. $\omega(X_H, \cdot) = dH$.

Indeed, we can determine X_H directly:

$$dH = \sum_{i=1}^{n} \sum_{j=1}^{3} \left(\frac{\partial H}{\partial p_{ij}} dp_{ij} + \frac{\partial H}{\partial q_{ij}} dq_{ij} \right)$$

OTOH, the contraction $\iota_{X_H} \omega = \omega(X_H, \cdot)$ has the form

$$\omega(X_H, \cdot) = \left(\sum_{i=1}^{n} \sum_{j=1}^{3} dq_{ij} \wedge dp_{ij} \right) (X_H, \cdot)$$

so we have to have $X_H = \sum_{i=1}^{n} \sum_{j=1}^{3} \left(\frac{\partial H}{\partial p_{ij}} \frac{\partial}{\partial q_{ij}} - \frac{\partial H}{\partial q_{ij}} \frac{\partial}{\partial p_{ij}} \right)$.

Now, what property does X_H have?

① If φ_t is the one-parameter family of diffeomorphisms generated by X_H, meaning $\varphi_0 = 1_M$ & $\overbrace{\frac{d\varphi_t}{dt} \circ \varphi_t^{-1} = X_H}^{\text{i.e. flow along } X_H}$

then $\varphi_t^* \omega = \omega$ $\forall t$. In other words, flowing along X_H preserves the symplectic structure.

② H is constant along trajectories of X_H (this is by Cartan's magic formula: $\mathcal{L}_{X_H} H = \iota_{X_H} dH = \iota_{X_H} \iota_{X_H} \omega = 0$)

③ In our particular case, $iX_H = \nabla H$. More precisely, let J be the std. complex structure

$J\left(\frac{\partial}{\partial q_{ij}}\right) = \frac{\partial}{\partial p_{ij}}$ & $J\left(\frac{\partial}{\partial p_{ij}}\right) = -\frac{\partial}{\partial q_{ij}}$. Then (w.r.t. the standard flat metric),

$$\nabla H = \sum_{i,j} \left(\frac{\partial H}{\partial q_{ij}} \frac{\partial}{\partial q_{ij}} + \frac{\partial H}{\partial p_{ij}} \frac{\partial}{\partial p_{ij}} \right) = JX_H.$$

But now, if $\rho(t) = (q(t), p(t))$ is a path in phase space, it satisfies the Hamilton equations $\begin{cases} \frac{dq_{ij}}{dt} = \frac{\partial H}{\partial p_{ij}} \\ \frac{dp_{ij}}{dt} = -\frac{\partial H}{\partial q_{ij}} \end{cases}$

if & only if $\rho(t)$ is a trajectory of X_H.

In particular, notice that we've now seen Newton's second law is equivalent to points in phase space following trajectories of Hamiltonian vector fields.

Also, this gives another proof of the conservation of energy.

In the next few classes, we'll develop some of the tools needed to make this all work (e.g., why is H constant along trajectories of X_H? Why do the φ_t preserve ω?)