Math 676: Day 1

The main object in this course: $(S^2)^n /\!\!/_{\vec{0}} SO(3)$.

What is this thing?

It is the symplectic reduction of the toric symplectic manifold $\underbrace{S^2 \times S^2 \times \ldots \times S^2}_{n}$ by the diagonal Hamiltonian action of $SO(3)$, reduced at the fiber over $\vec{0}$ in the moment polytope of the action.

It is (almost) a toric symplectic manifold, meaning it has action-angle coordinates & a moment map $\mu : (S^2)^n /\!\!/_{\vec{0}} SO(3) \longrightarrow P \subseteq \mathbb{R}^{n-3}$ where the pushforward measure on the convex polytope P is a constant multiple of Lebesgue measure.

The main goal of the course is to understand what the heck is going on in the previous paragraph.

Why should you care?

① This space turns out to be the moduli space of n-step closed random walks (a.k.a. random polygons) in $\mathbb{R}^3$, which provide the basic theoretical model for ring polymers like bacterial DNA.

Applied mathematicians want to be able to integrate over this space and to sample points uniformly from it (which is because sometimes the only way to integrate is to do Monte Carlo integration).

This area has a lot of impressive numerical experiments & heuristic/physics "proofs," but relatively little rigorous mathematical justification or real theorems.

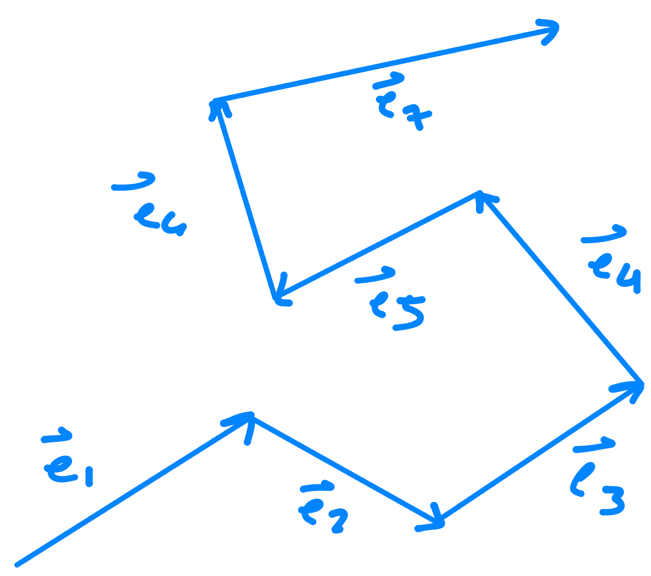
Much of what is rigorous in the field has been proved using symplectic geometry.

② Symplectic reduction is the natural quotient operation in the symplectic category. By the Kirwan-Kempf-Ness theorem, there is a correspondence (at least topologically) to the Geometric Invariant Theory (GIT) quotient for projective varieties. In particular, our space $(S^2)^n /\!/_{\vec{r}} SO(3)$ corresponds to the GIT quotient $(\mathbb{P}^1)^n /\!/_{\omega} PGL(2,\mathbb{C})$, which is a particular choice of compactification of the moduli space of n-pointed curves of genus 0, a.k.a. $\mathcal{M}_{0,n}$.

The connection b/w (algebro-geometric) moduli spaces & random walks seems to be completely unexplored, which obviously presents an opportunity for the motivated student.

Random walks

A classical random walk in $\mathbb{R}^d$ is an ordered collection of unit segments whose directions are chosen independently & uniformly on the unit sphere $S^{d-1} \subseteq \mathbb{R}^d$



For modeling polymers, we are interested in the shape of these walks & not so much their location, so it's convenient to take the quotient by the translation group $\mathbb{R}^d$. Equivalently, think of the edges as vectors.

Then the configuration space or moduli space of n-step (equilateral) random walks in $\mathbb{R}^d$ is just $\underbrace{S^{d-1} \times \ldots \times S^{d-1}}_{n}$.

This is a compact, Riemannian, $n(d-1)$-dimensional manifold. It is very simple to integrate over this space (use spherical coordinates!) & to sample points uniformly from it (how?).

In fact, in practice one usually doesn't care about the orientation of the random walk either, so the space you're really interested in is often $(S^{d-1})^n / SO(d)$, the configuration/moduli space of n-step random walks in $\mathbb{R}^d$ up to translation & rotation.

However:

① It's harder to come up w/ nice coords. on this space &

② This is not actually a manifold (why?)

So usually just deal w/ $(S^{d-1})^n$.

Closed walks / polygons

Suppose we now require our n-step random walks to form loops, meaning they return to their starting point after exactly n steps. This is obviously some subset of $(S^{d-1})^n$ or $(S^{d-1})/SO(d)$, but how can we get our hands on it?

What is the closure condition algebraically?

$\vec{e}_1 + \dots + \vec{e}_n = \vec{0}$. Notice that this is really d equations, one for each component of the vector.

To make it fancy, define $\mu : S^{d-1} \to \mathbb{R}^d$ by $\mu(\vec{e}_1, \dots, \vec{e}_n) = \vec{e}_1 + \dots + \vec{e}_n$. Then the loop random walks are $\mu^{-1}(\vec{0})$. If $\vec{0}$ is a regular value of μ, then this should be a smooth submanifold of $(S^{d-1})^n$ of dimension $n(d-1) - d = nd - n - d$.

We might also be interested in $\mu^{-1}(\vec{0})/SO(d)$ which should be a manifold (or something) of dimension

$$nd - n - d - \dim(SO(d)) = nd - n - d - \frac{d(d-1)}{2} = n(d-1) - \frac{d(d+1)}{2}$$

For polymers we are mostly interested in the case $d = 3$, in which case these are the spaces & their (real) dimensions:

space	dimension
$(S^2)^n$	$2n$
$(S^2)^n/SO(3)$	$2n-3$
$\mu^{-1}(\vec{0})$	$2n-3$
$\mu^{-1}(\vec{0})/SO(3)$	$2n-6$

Taking for granted the connection to (complex) algebraic geometry, your attention should naturally focus on the spaces with even real dimensions, since these at least have a chance of being complex algebraic varieties.

Indeed, since $S^2 \cong \mathbb{CP}^1$, we have a natural identification b/w the moduli space of random walks & the projective variety $(\mathbb{P}^1)^n$.

Much more subtle is the fact that $\mu^{-1}(\vec{0})/SO(3)$ is also a projective variety.

Before we get there, tho, we need to talk a bit about manifolds...