Network Algorithms

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Outline

Typical Network Problems

- Minimum Cost Spanning Tree
- All Pairs Shortest Distance / Paths
- Maximum Network Flow
- Travelling Sales Person (TSP)
- Graph Clustering

Combinatorial Optimization

- What is an Optimization Problem?
- What is a Global Optimum?
- What is a Local Optimum?
- Lin-Kernighan 2-opt
- Local Search
- Application: The Domino Portrait Problem

Abstract

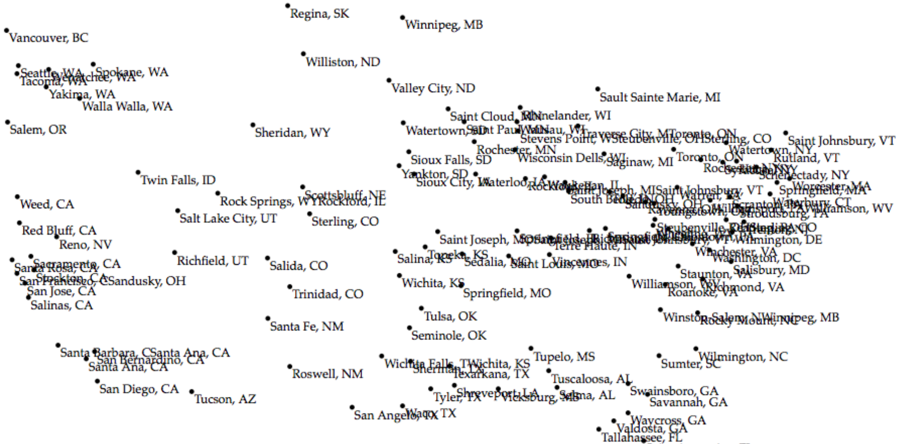
The talk presents some ideas on how combinatorial optimization can be used to design efficient algorithms for graphs and networks. Local Search is a relatively simple method which was proven to be effective in many areas, for instance graph clustering problems.

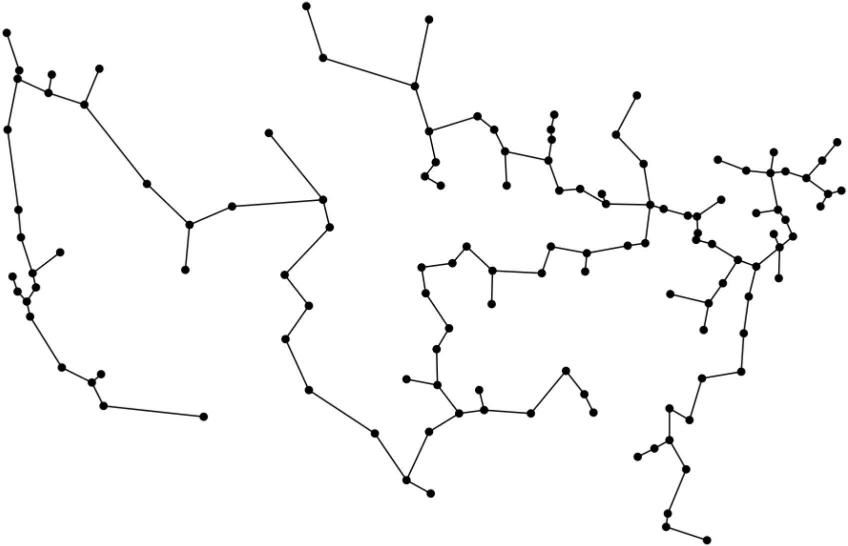
Typical Network Problems:

1. Minimum Cost Spanning Tree
2. All Pairs Shortest Distance / Paths
3. Maximum Network Flow (between two nodes, called source and target)
4. Travelling Sales Person (TSP)
5. Clustering

Minimum Cost Spanning Tree:

Find the cheapest spanning tree
(“connect the dots without creating cycles”)





Algorithms by Kruskal and by Prim, very effective:

“Add the cheapest edge which is still possible until everything is connected.”

Movie 1

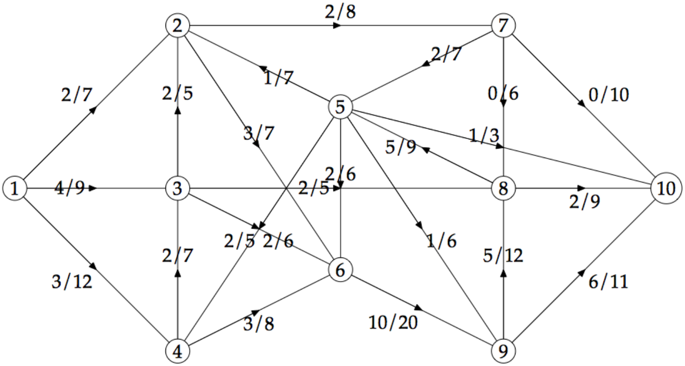
All Pairs Shortest Distance / Paths

Dijkstra's algorithm

Maximum Network Flow

Algorithm of Ford and Fulkerson:

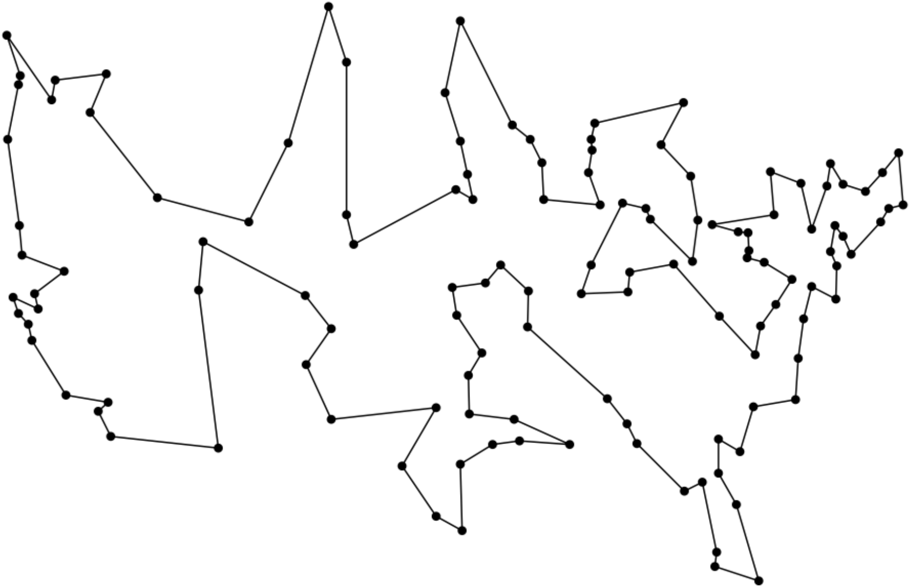
"augment the current flows until no augmenting path can be found anymore"



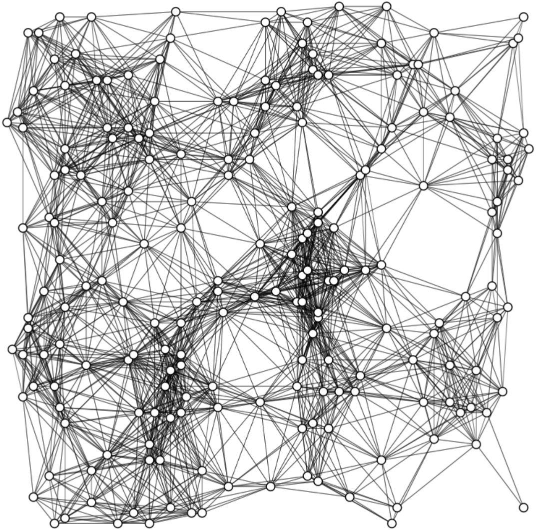
Travelling Salesman

Visit all cities on a cyclic tour.

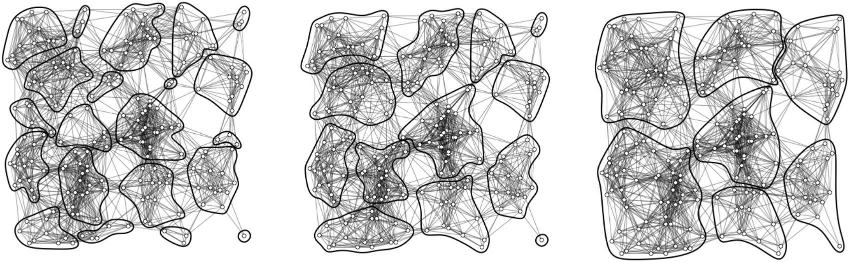
Algorithm: Lin / Kernighan: “2-opt”



Graph Clustering:



find "clusters", for instance:



Example taken from
Andrew King: *Graph Clustering with Restricted Neighbourhood Search,* Ph.D. thesis, University of Toronto, Department of Computer Science.

What is an Optimization Problem?

An instance of an optimization problem is a pair (F, c) where

1. F is a set, whose elements are called “feasible solutions”
2. $c : F \to \mathbb{R}$ a cost function

What is a Global Optimum?

An element $x \in F$ with $c(x) \leq c(y)$ for all $y \in F$ is called a global optimum.

It is often too hard determine a global optimum.

The set F may be prohibitively large.

What is a Local Optimum?

An element $x \in F$ with $c(x) \leq c(y)$ for all $y \in F$ which are "close" to x is called local optimum.

For this to make sense, one needs to define a concept of "neighborhood"

This is problem dependent.

Sometimes it is hard to find even a single point in F.

Lin-Kernighan 2-opt

Let F be the set of all tours, i.e. sequences

$[i_1, i_2, \ldots, i_n]$ which are permutations of the cities $1, \ldots, n$

the tour is $i_1 - i_2 - i_3 - \cdots - i_n - i_1$ (cyclically).

The size of F is prohibitively large

Lin-Kernighan 2-opt

Let

$$c([i_1, i_2, \ldots, i_n]) = \sum_{j=1}^{n-1} \mathrm{dist}(i_j, i_{j+1}) + \mathrm{dist}(i_n, i_1),$$

i.e., the cost of a tour is the sum of the distances travelled.

Lin-Kernighan 2-opt

Start with a random tour $x \in F$,

$x = [i_1, i_2, \ldots, i_n]$

i.e.,

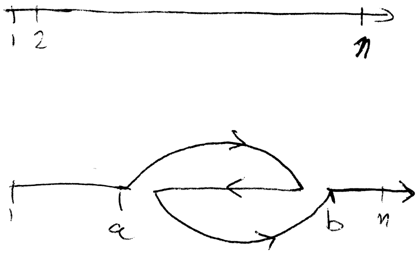
$$i_1 \mapsto i_2 \mapsto \quad \cdots \quad \mapsto i_n \mapsto i_1$$

Lin-Kernighan 2-opt

Choose two random numbers a, b with $1 \leq a \leq b \leq n$ and a, b at cyclic distance ≥ 2.

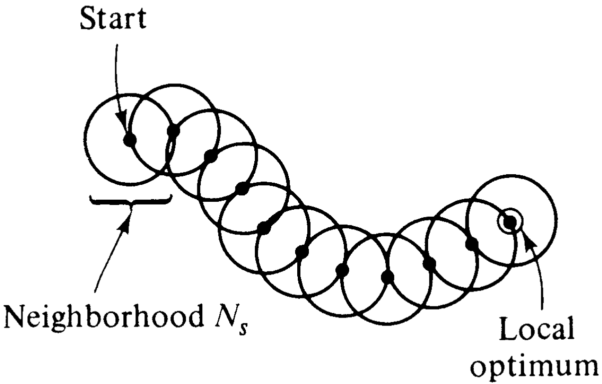
Reconnect as follows:

$$i_1 \mapsto \cdots \mapsto i_a \mapsto i_{b-1} \mapsto i_{b-2} \mapsto \cdots \mapsto i_{a+1} \mapsto i_b \mapsto \cdots \mapsto i_n \mapsto i_1 .$$



Local Search

Idea:
given $x \in F$
we try to find an improvement in a “neighborhood” $N(x)$



Local Search: Variable Depth

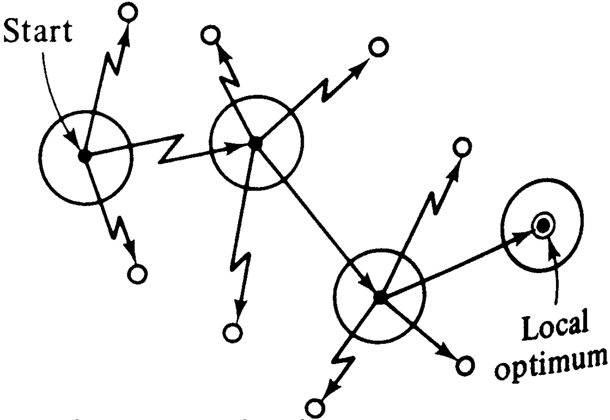
Idea:

do a (random) number of neighbor's neighbors.

Only the last of this chain of neighbors is compared to the current point.

The hope is that this allows us to go through a valley (or climb over a mountain)

Local Search: Variable Depth

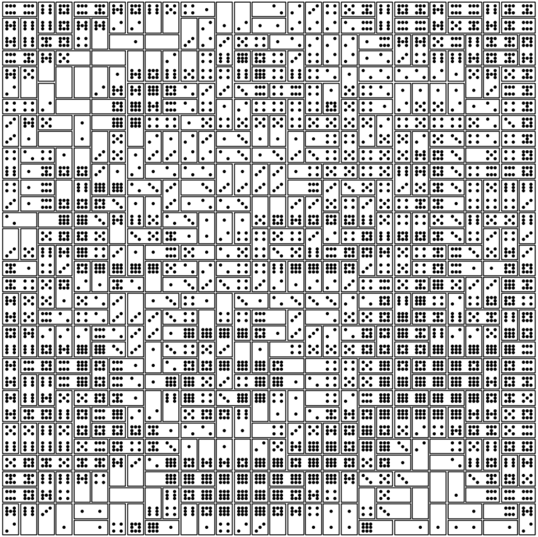


The last two pictures are taken from:
Papadimitriou/Steiglitz: Combinatorial Optimization.

Application: The Domino Portrait Problem

We want to approximate a foto portrait using Dominos (double nine, say).

We wish to use a fixed number of complete sets.



Bader Al-Shamarey: Two Topics in Combinatorial Optimization. Ph.D. thesis, CSU, 2007.