Decomposition algorithms in Clifford algebras

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[1] Joint work with David Wylie

$\mathcal{C}\ell_Q(V)$

- Vector space V over $\mathbb{R}$. $n = \mathbf{dim}(V)$.
- Nondegenerate quadratic form Q, inducing $\langle\cdot,\cdot\rangle_Q$.
- 2^n dim'l algebra obtained by associative linear extension of

$$\mathbf{x}\mathbf{y} = \langle\mathbf{x},\mathbf{y}\rangle_Q + \mathbf{x}\wedge\mathbf{y}$$

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$$\mathbf{xy} = \langle \mathbf{x}, \mathbf{y} \rangle_Q + \mathbf{x} \wedge \mathbf{y}$$

- Familiar special cases: $\mathbb{C}$, $\mathbb{H}$.

Example: Clifford algebra $\mathcal{C}\ell_{p,q}$

1. Real, associative algebra of dimension 2^n.
2. Generators $\{\mathbf{e}_i : 1 \leq i \leq n\}$ along with the unit scalar $\mathbf{e}_\emptyset = 1 \in \mathbb{R}$.
3. Generators satisfy:
   - $[\mathbf{e}_i, \mathbf{e}_j] := \mathbf{e}_i \, \mathbf{e}_j + \mathbf{e}_j \, \mathbf{e}_i = 0$ for $1 \leq i \neq j \leq n$,
   - $$\mathbf{e}_i{}^2 = \begin{cases} 1 & \text{if } 1 \leq i \leq p, \\ -1 & \text{if} p+1 \leq i \leq p+q. \end{cases}$$

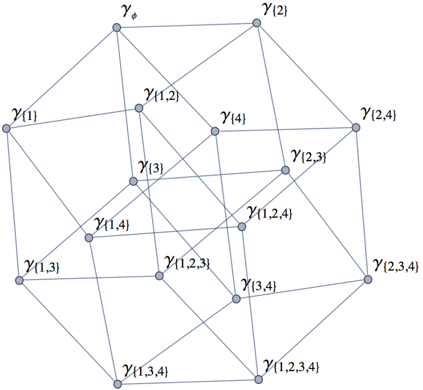
Multi-index notation

- Let $[n] = \{1, 2, \ldots, n\}$ and denote arbitrary, canonically ordered subsets of $[n]$ by capital Roman characters.
- $2^{[n]}$ denotes the *power set* of $[n]$.
- Elements indexed by subsets:

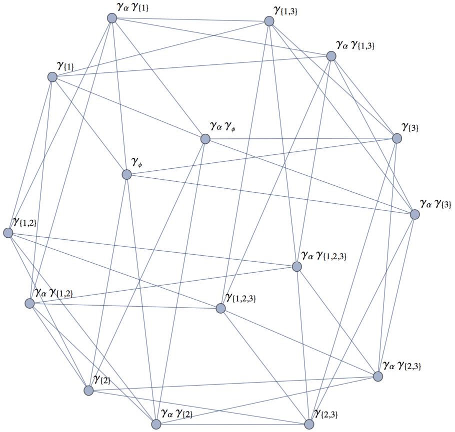
$$\gamma_J = \prod_{j \in J} \gamma_j.$$

- Natural binary representation
- Generators: any orthonormal basis for V.

Hypercube $\mathcal{Q}_4$



Modified hypercubes



Example: $\mathcal{C}\ell_{n,0}$ (a.k.a. $\mathcal{C}\ell_n$)

- $[n] = \{1, 2, \ldots, n\}$
- For $j \in [n]$ and $I \subset [n]$, define $\mu_j(I) = \sharp\{i \in I : i > J\}$
  counting measure
- $I \triangle J = (I \cup J) \setminus (I \cap J)$
  set symmetric difference
- Basis blade multiplication:

$$\mathbf{e}_I \mathbf{e}_J = (-1)^{\sum_{j \in J} \mu_j(I)} \mathbf{e}_{I \triangle J}$$

A randomly-generated grade-5 element of $\mathcal{C}\ell_8$

$$
\begin{aligned}
&6784\,e_{\{1\}} + 6678\,e_{\{2\}} + 10\,984\,e_{\{3\}} + 7576\,e_{\{4\}} - 6205\,e_{\{5\}} - 102\,e_{\{6\}} - 5149\,e_{\{7\}} + 1202\,e_{\{8\}} - 676\,e_{\{1,2,3\}} - 1752\,e_{\{1,2,4\}} +\\
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\end{aligned}
$$

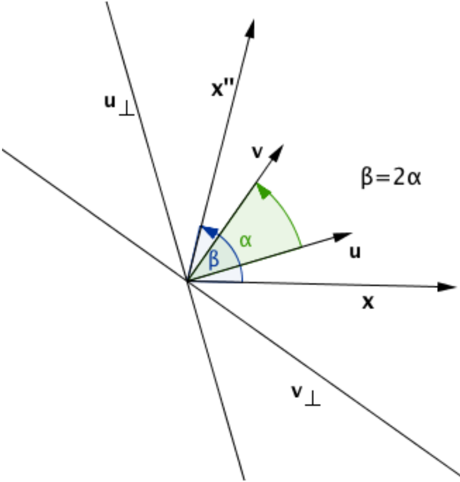
Geometric algebra

Consider the Euclidean signature.

- Quadratic form is positive definite; generators satisfy ${\mathbf{e}_i}^2 = 1$.
- Given unit vector $\mathbf{u} \in \mathbb{R}^2$, $\mathbf{x} \mapsto -\mathbf{uxu}$ represents reflection across hyperplane $\mathbf{u}^\perp$.
- Given unit vectors $\mathbf{u}, \mathbf{v}$ with $\angle\mathbf{uv} = \alpha$, $\mathbf{x} \mapsto \mathbf{vuxuv}$ is 2α rotation in $\mathbf{uv}$-plane.

Motivation: the Euclidean case

$x \mapsto vuxuv$



Automorphisms

Arbitrary element $u = \sum_{I \in 2^{[n]}} u_I \mathbf{e}_I$.

- Grade involution: $\hat{u} = \sum_{I \in 2^{[n]}} (-1)^{|I|} u_I \mathbf{e}_I$.
- Reversion: $\tilde{u} = \sum_{I \in 2^{[n]}} (-1)^{\frac{|I|(|I|-1)}{2}} u_I \mathbf{e}_I$.
- Clifford Conjugate: $\overline{u} = \sum_{I \in 2^{[n]}} (-1)^{\frac{|I|(|I|+1)}{2}} u_I \mathbf{e}_I$.

The conformal orthogonal group $\mathrm{CO}_Q(V)$

- Q is a nondegenerate quadratic form.
- V is an n-dimensional $\mathbb{R}$-vector space with inner product $\langle , \rangle_Q$ induced by Q.
- $\mathcal{C}\ell_Q(V)$ is the Clifford algebra of this space.
- Conformal orthogonal group $\mathrm{CO}_Q(V)$ is the direct product of dilations and Q-orthogonal linear transformations of V.

Hyperplane Reflections in V

1. Product of orthogonal vectors is a *blade*.
2. Given unit blade $\mathfrak{u} \in \mathcal{C}\ell_Q(V)$, where Q is positive definite.
3. The map $\mathbf{x} \mapsto \mathfrak{u}\mathbf{x}\widehat{\mathfrak{u}^{-1}}$ represents a composition of orthogonal hyperplane reflections.
4. Each vertex of the hypercube underlying the Cayley graph corresponds to a hyperplane arrangement.

Historical Framework

- Versor factorization algorithms: Christian Perwass [2].
- Efficient blade factorization algorithms: Dorst and Fontijne [3], [4].

[2] *Geometric Algebra with Applications in Engineering*, Springer-Verlag, Berlin, 2009.

[3] L. Dorst, D. Fontijne, *Efficient Algorithms for Factorization and Join of Blades*, Geometric Algebra Computing, E. Bayro-Corrochano, G. Scheuermann, Eds., Springer, London, 2010, pp. 457–476.

[4] D. Fontijne, *Efficient Implementation of Geometric Algebra*, Ph.D. thesis, University of Amsterdam, 2007.

Helmstetter's work

Factorization in Clifford algebras of arbitrary signature was considered by Helmstetter [5].

- Lipschitz monoid: multiplicative monoid generated in $\mathcal{C}\ell_Q(V)$ over a field $\Bbbk$ by all scalars in $\Bbbk$, all vectors in V, and all $1 + \mathbf{xy}$ where $\mathbf{x}$ and $\mathbf{y}$ are vectors that span a totally isotropic plane.
- Elements called Lipschitzian elements.
- Given a Lipschitzian element a in $\mathcal{C}\ell_Q(V)$ over $\Bbbk$ containing at least three scalars.
- If a is not in the subalgebra generated by a totally isotropic subspace of V, then it is a product of linearly independent vectors of V.

[5] J. Helmstetter, Factorization of Lipschitzian elements, *Advances in Applied Clifford Algebras*, **24** (2014), 675–712.

Definition

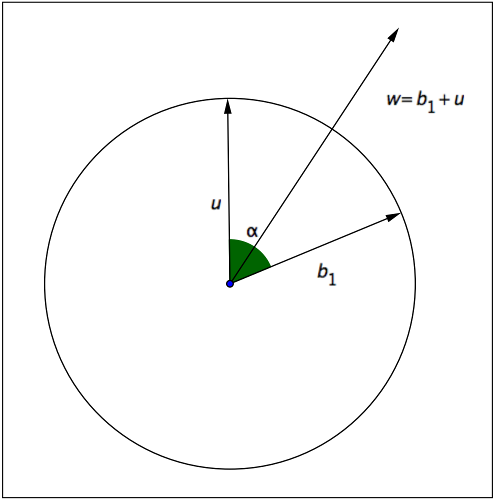
An invertible element $\mathfrak{u} \in \mathcal{C}\ell_Q(V)$ of grade k is said to be *decomposable* if there exists a linearly independent collection $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ of anisotropic vectors in V such that $\mathfrak{u} = \mathbf{w}_1 \cdots \mathbf{w}_k$ and $\langle \mathfrak{u} \rangle_k$ is invertible [a]. In this case, $\mathfrak{u}$ is referred to as a *decomposable* k-element.

[a] Requiring invertibility of the top form makes the *decomposable* elements of $\mathcal{C}\ell_Q(V)$ a proper subset of the Lipschitz group of $\mathcal{C}\ell_Q(V)$.

Motivation: the Euclidean case

Consider $\mathfrak{b} = 4 + 8\mathbf{e}_{\{1,2\}} + 6\mathbf{e}_{\{1,3\}} - 6\mathbf{e}_{\{2,3\}} \in \mathcal{C}\ell_3$. The action of $\mathbf{x} \mapsto \mathfrak{b}\mathbf{x}\overline{\mathfrak{b}}$ is the composition of a plane rotation and dilation by factor $\mathfrak{b}\tilde{\mathfrak{b}} = 152$ in $\mathbb{R}^3$. Letting $\mathbf{p} = \mathbf{e}_1$ serve as a "probing vector," we compute $\mathbf{p}' = \mathfrak{b}\mathbf{p}\widehat{\mathfrak{b}^{-1}}$ and obtain $\mathbf{p}' = -\frac{6}{19}\mathbf{e}_1 + \frac{1}{19}\mathbf{e}_2 - \frac{18}{19}\mathbf{e}_3$. Letting $\mathbf{b}_1 = (\mathbf{p} - \mathbf{p}')/\|\mathbf{p} - \mathbf{p}'\|$, we obtain the normalized projection $\mathbf{b}_1$ of $\mathbf{p}$ into the plane of rotation. In particular,

$$\mathbf{b}_1 = -\frac{5}{\sqrt{38}}\mathbf{e}_1 + \frac{1}{5\sqrt{38}}\mathbf{e}_2 - \frac{9\sqrt{2}}{5\sqrt{19}}\mathbf{e}_3.$$


w=b1+u
u
α
b1

Computing $\mathbf{u} = \mathfrak{b}\mathbf{b}_1\widehat{\mathfrak{b}^{-1}}$, we obtain

$$\mathbf{u} = \frac{275\mathbf{e}_1 + 293\mathbf{e}_2 + 426\mathbf{e}_3}{95\sqrt{38}}.$$

Computing the unit vector $\mathbf{b}_2$, which lies halfway between $\mathbf{b}_1$ and its image, we obtain

$$\mathbf{b}_2 = (\mathbf{b}_1 + \mathbf{u})/\|\mathbf{b}_1 + \mathbf{u}\| = -\frac{50}{95}\mathbf{e}_1 + \frac{78}{95}\mathbf{e}_2 + \frac{21}{95}\mathbf{e}_3,$$

The rotation induced by $\mathfrak{b}$ now corresponds to the composition of two reflections across the orthogonal complements of $\mathbf{b}_1$ and $\mathbf{b}_2$, respectively. Note that $\mathbf{b}_2$ is the normalization of $\mathbf{w}$ in the previous figure. The factorization of $\mathfrak{b}$ is then given by

$$\mathfrak{b} = \sqrt{152}\,\mathbf{b}_2\mathbf{b}_1 = 4 + 8\mathbf{e}_{\{1,2\}} + 6\mathbf{e}_{\{1,3\}} - 6\mathbf{e}_{\{2,3\}}.$$

Extending to non-Euclidean signatures

- Negative definite signatures.
- Indefinite signatures.

Extending to non-Euclidean signatures

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Extending to non-Euclidean signatures

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- Indefinite signatures. ☒ Be careful!

Existence Theorem

Given a decomposable k-element $\mathfrak{u} = \mathbf{w}_1 \cdots \mathbf{w}_k \in \mathcal{C}\ell_Q(V)$, let $n = \dim V$ and define $\varphi_{\mathfrak{u}} \in \mathrm{O}_Q(V)$ by

$$\varphi_{\mathfrak{u}}(\mathbf{v}) = \mathfrak{u}\mathbf{v}\widehat{\mathfrak{u}^{-1}}.$$

Then $\varphi_{\mathfrak{u}}$ has an eigenspace $\mathcal{E}$ of dimension $n-k$ with corresponding eigenvalue 1.

CliffordDecomp Algorithm

Input: $\mathfrak{b}$, a decomposable k-element.
Output: $\{\mathbf{b}_k, \ldots, \mathbf{b}_1\}$ such that $\mathfrak{b} = \mathbf{b}_k \cdots \mathbf{b}_1$.
$\ell \leftarrow 1$
$\mathfrak{u} \leftarrow \mathfrak{b}/\|\mathfrak{b}\|$
while $\sharp\mathfrak{u} > 1$ **do**
  Let $\mathbf{x} \in V$ such that $\mathbf{x} \lrcorner \mathfrak{u} \neq 0$ and $\mathbf{x}^2 \neq 0$ $\mathbf{x}' \leftarrow \mathfrak{u}\mathbf{x}\widehat{\mathfrak{u}^{-1}}$
  if $(\mathbf{x} - \mathbf{x}')^2 \neq 0$ **then**
    $\mathbf{b}_{\sharp\mathfrak{u}} \leftarrow (\mathbf{x} - \mathbf{x}')/\|\mathbf{x} - \mathbf{x}'\|$
    if $\mathfrak{u}\,{\mathbf{b}_{\sharp\mathfrak{u}}}^{-1}$ *is decomposable* **then**
      $\mathfrak{u} \leftarrow \mathfrak{u}\,{\mathbf{b}_{\sharp\mathfrak{u}}}^{-1}$
    end
  end
end
return $\{\mathbf{b}_k, \ldots, \mathbf{b}_2, \|\mathfrak{b}\|\mathfrak{u}\}$

A randomly-generated grade-5 element of $\mathcal{C}\ell_8$

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\begin{aligned}
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\end{aligned}
$$

Decomposition via CliffordDecomp

```
In[20]:= Timing[Bf = CliffordDecomp[B]]

Out[20]= {8.438983,
  {0.0630904 e{1} + 0.264971 e{2} - 0.390084 e{3} + 0.147402 e{4} - 0.188183 e{5} + 0.748969 e{6} + 0.136306 e{7} +
    0.370096 e{8}, 0.171923 e{1} - 0.380025 e{2} - 0.579994 e{3} + 0.143699 e{4} + 0.554634 e{5} - 0.320797 e{6} +
    0.193767 e{7} - 0.144584 e{8}, 0.423971 e{1} + 0.232783 e{2} + 0.30033 e{3} + 0.431616 e{4} -
    0.282986 e{5} + 0.31425 e{6} + 0.348015 e{7} - 0.435456 e{8}, 0.653833 e{1} + 0.148705 e{2} +
    0.184215 e{3} + 0.519516 e{4} + 0.304638 e{5} - 0.363147 e{6} - 0.137426 e{7} + 0.0546965 e{8},
   31906.1 e{1} + 10450.9 e{2} - 7684.13 e{3} - 3218.17 e{4} + 42554.1 e{5} + 3429.73 e{6} + 42580. e{7} - 14537.7 e{8}}}
```

Decomposing the 5-blade

```
In[24]:= Timing[Blf = CliffordDecomp[Bl]]

Out[24]= {2.055691, {0.436433 e{1} - 0.0787571 e{2} - 0.161208 e{3} +
            0.830435 e{4} - 0.225138 e{5} - 0.000738118 e{6} - 0.170457 e{7} - 0.0892718 e{8},
          0.180493 e{1} + 0.321771 e{2} + 0.856856 e{3} - 0.219418 e{5} - 0.192729 e{6} - 0.113192 e{7} - 0.177712 e{8},
          0.486322 e{1} + 0.110234 e{2} + 0.752037 e{5} - 0.391758 e{6} + 0.154873 e{7} + 0.0912149 e{8},
          0.246575 e{1} + 0.546527 e{2} + 0.621213 e{6} + 0.119359 e{7} + 0.490262 e{8},
          -5505.05 e{1} - 11762.3 e{6} - 26704.6 e{7} + 24174.3 e{8}}}
```

FastBladeFactor

- Multi indices are well ordered by $\mathbf{f}_I \preceq \mathbf{f}_J \Leftrightarrow \sum_{i \in I} 2^{i-1} \leq \sum_{j \in J} 2^{j-1}$.
- Define

$$\mathrm{FirstTerm}\left(\sum_I \alpha_I \mathbf{f}_I\right) := \min_{\{\mathbf{f}_X : \alpha_X \neq 0\}} \alpha_X \mathbf{f}_X.$$

FastBladeFactor

Input: Blade $\mathfrak{b} \in \mathcal{C}\ell_Q(V)$ of grade k as a sum $\sum_I \alpha_I \mathbf{f}_I$.
Output: Scalar α and set of vectors $\{\mathbf{b}_1, \ldots, \mathbf{b}_k\}$ such that $\mathfrak{b} = \alpha \mathbf{b}_k \wedge \cdots \wedge \mathbf{b}_1$.

$\alpha_M \mathbf{f}_M \leftarrow$FirstTerm($\mathfrak{b}$)   Say $M = \{m_1, \ldots, m_k\}$.

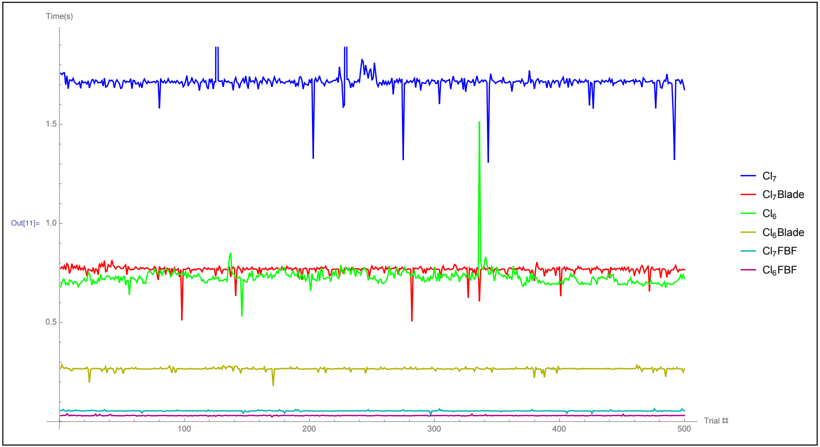
for $\ell \leftarrow 1$ **to** k **do**
  $\mathfrak{u} \leftarrow \mathbf{f}_{M \setminus \{m_\ell\}}$
  $\mathbf{b}_\ell \leftarrow \langle \mathfrak{b}\mathfrak{u}^{-1} \rangle_1$
end
return $\{\alpha_M, \mathbf{b}_1, \ldots, \mathbf{b}_k\}$

. . . via FastBladeFactor

```
In[26]:= Timing[Blff = FastBladeFactor[Bl]]
```

Out[26]= $\left\{0.107119, \left\{e_{\{1\}} + \frac{344\, e_{\{6\}}}{161} + \frac{781\, e_{\{7\}}}{161} - \frac{101\, e_{\{8\}}}{23}, -e_{\{2\}} - \frac{139\, e_{\{6\}}}{805} + \frac{1586\, e_{\{7\}}}{805} - \frac{331\, e_{\{8\}}}{115}, e_{\{3\}} - \frac{993\, e_{\{6\}}}{805} - \frac{878\, e_{\{7\}}}{805} + \frac{33\, e_{\{8\}}}{115}, -e_{\{4\}} + \frac{43\, e_{\{6\}}}{23} + \frac{89\, e_{\{7\}}}{23} - \frac{74\, e_{\{8\}}}{23}, e_{\{5\}} - \frac{1552\, e_{\{6\}}}{805} - \frac{2127\, e_{\{7\}}}{805} + \frac{292\, e_{\{8\}}}{115}, -1610\right\}\right\}$

4-elements and 4-blades in $\mathcal{C}\ell_6$ and $\mathcal{C}\ell_7$.



Complexity of representations

- Blades are “simple” compared to more general elements.
- FBF is fast.
- Number of terms in canonical expansion is understood.
- Can we do better?

Lemma (Wylie)

If $\mathfrak{v} \in \mathcal{C}\ell_Q(V)$ is a decomposable k-element for $k \leq \dim V$, then $c_{k,j}$, as defined below, gives an upper bound on the number of blades required to express $\langle \mathfrak{v} \rangle_j$ as a sum of blades. This upper bound satisfies the following recurrence:

$$c_{k,j} = \begin{cases} \frac{(-1)^{k-j}+1}{2} & \text{if } j = 0 \text{ or } 1 \\ 1 & \text{if } j = k \\ c_{k-1,j-1} + c_{k-1,j+1} & \text{if } 1 < j < k \\ 0 & \text{if } j > k \end{cases}$$

$k \backslash j$	0	1	2	3	4	5	6	7	8	9	10	T_k
1		1										1
2	1		1									2
3		1		1								2
4	1		2		1							4
5		1		3		1						5
6	1		4		4		1					10
7		1		8		5		1				15
8	1		9		13		6		1			30
9		1		22		19		7		1		50
10	1		23		41		26		8		1	100

Table: Values of $c_{k,j}$

Open questions & avenues for further research

- How to *efficiently* break up into blades?
- Improvement by simple change of basis?
- Decompositions in "combinatorially interesting" Clifford subalgebras?
- Near-blade approximations?

Blade conjugation

1. $\mathfrak{u} \in \mathcal{C}\ell_Q(V)$ a blade.
2. $\Phi_{\mathfrak{u}}(\mathbf{x}) := \mathfrak{u}\mathbf{x}\widehat{\mathfrak{u}^{-1}}$ is a Q-orthogonal transformation on V.
3. The operators are self-adjoint w.r.t. $\langle \cdot, \cdot \rangle_Q$; i.e., they are *quantum random variables*.
4. Characteristic polynomial of $\Phi_{\mathfrak{u}}$ generates *Kravchuk polynomials*:

$$\chi(t) = (t + (-1)^k)^k (t - (-1)^k)^{n-k}$$

Induced operators

1. $\Phi_{\mathfrak{u}}$ induces $\varphi_{\mathfrak{u}}$ on $\mathcal{C}\ell_Q(V)$.
2. Conjugation operators allow decomposition of blades.
    - Eigenvalues ± 1
    - Basis for each eigenspace provides factorization of corresponding blade.
3. Quantum random variables obtained at every level of induced operators.
    - $\varphi^{(\ell)}$ is self-adjoint w.r.t. Q-inner product for each $\ell = 1, \ldots, n$.
4. Kravchuk polynomials appear in traces at every level.
5. Kravchuk matrices represent blade conjugation operators (in most cases[6]).

[6] G.S. Staples, Kravchuk Polynomials & Induced/Reduced Operators on Clifford Algebras, *Complex Analysis and Operator Theory* (2014).

Idea: Graph-Induced Operators

- Let A be the adjacency matrix or combinatorial Laplacian of a graph.
- View A as a linear operator on V.
- A naturally induces an operator $\mathfrak{A}$ on the Clifford algebra $\mathcal{C}\ell_Q(V)$ according to action (multiplication, conjugation, etc.) on $\mathcal{C}\ell_Q(V)$.

Idea: Graph-Induced Operators

- Particular subalgebras: fermions and zeons.
- Induced operators reveal information about the graph.
- Enumeration of Hamiltonian cycles and spanning trees[7].

[7] G.S. Staples. Graph-induced operators: Hamiltonian cycle enumeration via fermion-zeon convolution. *Preprint*.

Operator Calculus (OC)

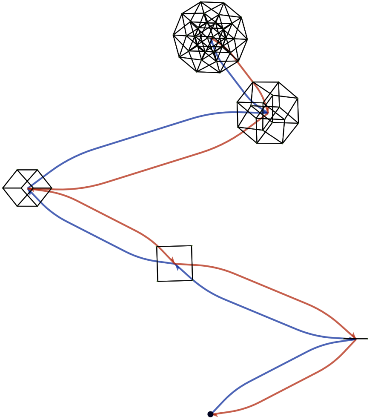
1. Lowering operator Λ
   - differentiation
   - annihilation
   - deletion
2. Raising operator Ξ
   - integration
   - creation
   - addition/insertion

OC & Clifford multiplication

1. Left lowering $\Lambda_{\mathbf{x}}$: $u \mapsto \mathbf{x} \lrcorner u$
2. Right lowering $\check{\Lambda}_{\mathbf{x}}$: $u \mapsto u \llcorner \mathbf{x}$
3. Left raising $\Xi_{\mathbf{x}}$: $u \mapsto \mathbf{x} \wedge u$
4. Right raising $\check{\Xi}_{\mathbf{x}}$: $u \mapsto u \wedge \mathbf{x}$
5. Clifford product satisfies

$$\mathbf{x}u = \Lambda_{\mathbf{x}}\mathbf{u} + \Xi_{\mathbf{x}}u$$
$$u\mathbf{x} = \check{\Lambda}_{\mathbf{x}}\mathbf{u} + \check{\Xi}_{\mathbf{x}}u$$

Raising & Lowering



OC on Graphs - Schott & Staples

- Walks on hypercubes
- Walks on Clifford algebras (homogeneous and dynamic)
- Graph enumeration problems
- Routing in networks
- More [8]

[8] R. Schott, G.S. Staples, *Operator Calculus on Graphs (Theory and Applications in Computer Science)*, Imperial College Press, London, 2012.

To be continued...

THANKS FOR YOUR ATTENTION!

Selected readings

- G.S. Staples, D. Wylie. Clifford algebra decompositions of conformal orthogonal group elements. *Preprint*, 2015.
- C. Cassiday, G. S. Staples. On representations of semigroups having hypercube-like Cayley graphs. *Clifford Analysis, Clifford Algebras and Their Applications*, **4** (2015), 111-130.
- G.S. Staples, Kravchuk polynomials & induced/reduced operators on Clifford algebras, *Complex Analysis and Operator Theory* (2014), `dx.doi.org/10.1007/s11785-014-0377-z`.
- G. Harris, G.S. Staples. Spinorial formulations of graph problems, *Advances in Applied Clifford Algebras*, **22** (2012), 59–77.

More on Clifford algebras, operator calculus, and graph theory

- R. Schott, G.S. Staples, *Operator Calculus on Graphs (Theory and Applications in Computer Science)*, Imperial College Press, London, 2012.