# Classifying cocyclic Butson Hadamard matrices

Ronan Egan*

Joint work with Dane Flannery and Padraig Ó Catháin.

July 23, CoCoA 2015.

*Supported by an NUI Galway Hardiman Scholarship and the Irish Research council

# Introduction

Let $\zeta_k = e^{\frac{2\pi i}{k}}$. An $n \times n$ matrix $H$ with entries in $\langle \zeta_k \rangle$ is a $\mathrm{BH}(n, k)$ (*Butson Hadamard matrix*)

# Introduction

Let $\zeta_k = e^{\frac{2\pi i}{k}}$. An $n \times n$ matrix $H$ with entries in $\langle \zeta_k \rangle$ is a $\mathrm{BH}(n,k)$ (*Butson Hadamard matrix*) if

$$HH^* = nI_n$$

where $*$ denotes Hermitian transpose,

# Introduction

Let $\zeta_k = e^{\frac{2\pi i}{k}}$. An $n \times n$ matrix $H$ with entries in $\langle \zeta_k \rangle$ is a $\mathrm{BH}(n, k)$ (*Butson Hadamard matrix*) if

$$HH^* = nI_n$$

where $*$ denotes Hermitian transpose, i.e., $H = [h_{ij}] \Rightarrow H^* = [h_{ji}^{-1}]$.

# Introduction

Let $\zeta_k = e^{\frac{2\pi i}{k}}$. An $n \times n$ matrix $H$ with entries in $\langle \zeta_k \rangle$ is a $\mathrm{BH}(n,k)$ (*Butson Hadamard matrix*) if

$$HH^* = nI_n$$

where $*$ denotes Hermitian transpose, i.e., $H = [h_{ij}] \Rightarrow H^* = [h_{ji}^{-1}]$.

**Example**

$$H = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \zeta_3 & \zeta_3^2 \\ 1 & \zeta_3^2 & \zeta_3 \end{bmatrix}$$ is a $\mathrm{BH}(3,3)$.

# Introduction

Let $\zeta_k = e^{\frac{2\pi i}{k}}$. An $n \times n$ matrix $H$ with entries in $\langle \zeta_k \rangle$ is a $\mathrm{BH}(n, k)$ (*Butson Hadamard matrix*) if

$$HH^* = nI_n$$

where $*$ denotes Hermitian transpose, i.e., $H = [h_{ij}] \Rightarrow H^* = [h_{ji}^{-1}]$.

**Example**

$$H = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \zeta_3 & \zeta_3^2 \\ 1 & \zeta_3^2 & \zeta_3 \end{bmatrix} \text{ is a } \mathrm{BH}(3,3).$$

If $k$ is prime, then $k$ must divide $n$. If $k = 2$, $H$ is a real Hadamard matrix, and for $n > 2$ it is known that $n$ is a multiple of 4.

# Motivation

Butson matrices play a crucial role in Quantum Information Theory. A library of known Complex Hadamard matrices is kept up to date at the URL below.

http://chaos.if.uj.edu.pl/∼karol/hadamard/

# Motivation

Butson matrices play a crucial role in Quantum Information Theory. A library of known Complex Hadamard matrices is kept up to date at the URL below.

`http://chaos.if.uj.edu.pl/∼karol/hadamard/`

Real Hadamard matrices have been extensively studied, but Butsons have not.

# Motivation

Butson matrices play a crucial role in Quantum Information Theory. A library of known Complex Hadamard matrices is kept up to date at the URL below.

`http://chaos.if.uj.edu.pl/∼karol/hadamard/`

Real Hadamard matrices have been extensively studied, but Butsons have not. We classify up to equivalence, all cocyclic $\mathrm{BH}(n, p)$s for odd prime $p$, and $np \leq 100$. This includes generating previously unknown Butson matrices.

# Equivalence relations

Let $X$ and $Y$ be $\mathrm{BH}(n, k)$s. We say $X \approx Y$ are *equivalent* if $MXN = Y$ for monomial matrices $M$ and $N$ with non-zero entries in $\langle \zeta_k \rangle$.

# Equivalence relations

Let $X$ and $Y$ be $\mathrm{BH}(n,k)$s. We say $X \approx Y$ are *equivalent* if $MXN = Y$ for monomial matrices $M$ and $N$ with non-zero entries in $\langle \zeta_k \rangle$. If $M, N \in \mathrm{Perm}(n)$, then $X \sim Y$ are *permutation equivalent*.

# Equivalence relations

Let $X$ and $Y$ be $\mathrm{BH}(n,k)$s. We say $X \approx Y$ are *equivalent* if $MXN = Y$ for monomial matrices $M$ and $N$ with non-zero entries in $\langle \zeta_k \rangle$. If $M, N \in \mathrm{Perm}(n)$, then $X \sim Y$ are *permutation equivalent*.

The direct product $\mathrm{Mon}(n, \langle \zeta_k \rangle) \times \mathrm{Mon}(n, \langle \zeta_k \rangle)$ acts on $\mathrm{BH}(n,k)$ via $(M,N)X = MXN^*$.

- The orbit of $X$ under this action is its equivalence class.
- The stabilizer of $X$ is its full *automorphism group* $\mathrm{Aut}(X)$.

# Equivalence relations

Let $X$ and $Y$ be $\mathrm{BH}(n,k)$s. We say $X \approx Y$ are *equivalent* if $MXN = Y$ for monomial matrices $M$ and $N$ with non-zero entries in $\langle \zeta_k \rangle$. If $M, N \in \mathrm{Perm}(n)$, then $X \sim Y$ are *permutation equivalent*.

The direct product $\mathrm{Mon}(n, \langle \zeta_k \rangle) \times \mathrm{Mon}(n, \langle \zeta_k \rangle)$ acts on $\mathrm{BH}(n,k)$ via $(M,N)X = MXN^*$.

- The orbit of $X$ under this action is its equivalence class.
- The stabilizer of $X$ is its full *automorphism group* $\mathrm{Aut}(X)$.

The subgroup of $\mathrm{Aut}(X)$ comprised of pairs of permutation matrices, is denoted by $\mathrm{PermAut}(X)$.

# Cocyclic matrices

A matrix $M \in \mathrm{Mat}(n, \langle\zeta_k\rangle)$ is *cocyclic* over a group $G$ if $M \approx [\psi(g,h)]_{g,h\in G}$ for some 2-cocycle $\psi : G \times G \to \langle\zeta_k\rangle$

# Cocyclic matrices

A matrix $M \in \mathrm{Mat}(n, \langle\zeta_k\rangle)$ is *cocyclic* over a group $G$ if $M \approx [\psi(g,h)]_{g,h\in G}$ for some 2-cocycle $\psi : G \times G \to \langle\zeta_k\rangle$ i.e.,

$$\psi(g,h)\psi(gh,f) = \psi(g,hf)\psi(h,f), \ \forall\, g,h,f \in G.$$

# Cocyclic matrices

A matrix $M \in \mathrm{Mat}(n, \langle\zeta_k\rangle)$ is *cocyclic* over a group $G$ if $M \approx [\psi(g,h)]_{g,h\in G}$ for some 2-cocycle $\psi : G \times G \to \langle\zeta_k\rangle$ i.e.,

$$\psi(g,h)\psi(gh,f) = \psi(g,hf)\psi(h,f), \ \forall\ g,h,f \in G.$$

$\psi$ is *normalized* if $\psi(g,1) = \psi(1,h) = 1$, for all $g,h \in G$,

# Cocyclic matrices

A matrix $M \in \mathrm{Mat}(n, \langle \zeta_k \rangle)$ is *cocyclic* over a group $G$ if $M \approx [\psi(g,h)]_{g,h \in G}$ for some 2-cocycle $\psi : G \times G \to \langle \zeta_k \rangle$ i.e.,

$$\psi(g,h)\psi(gh,f) = \psi(g,hf)\psi(h,f), \ \forall \ g,h,f \in G.$$

$\psi$ is *normalized* if $\psi(g,1) = \psi(1,h) = 1$, for all $g, h \in G$, and we say $M$ is *row/column balanced* if each element of $\langle \zeta_k \rangle$ appears equally often in every non-initial row/column of $M$.

# Cocyclic matrices

A matrix $M \in \mathrm{Mat}(n, \langle \zeta_k \rangle)$ is *cocyclic* over a group $G$ if $M \approx [\psi(g,h)]_{g,h \in G}$ for some 2-cocycle $\psi : G \times G \to \langle \zeta_k \rangle$ i.e.,

$$\psi(g,h)\psi(gh,f) = \psi(g,hf)\psi(h,f), \ \forall \ g,h,f \in G.$$

$\psi$ is *normalized* if $\psi(g,1) = \psi(1,h) = 1$, for all $g,h \in G$, and we say $M$ is *row/column balanced* if each element of $\langle \zeta_k \rangle$ appears equally often in every non-initial row/column of $M$.

In this case, $\psi$ is *orthogonal*.

# Cocyclic matrices

A matrix $M \in \mathrm{Mat}(n, \langle\zeta_k\rangle)$ is *cocyclic* over a group $G$ if $M \approx [\psi(g,h)]_{g,h\in G}$ for some 2-cocycle $\psi : G \times G \to \langle\zeta_k\rangle$ i.e.,

$$\psi(g,h)\psi(gh,f) = \psi(g,hf)\psi(h,f), \ \forall\ g,h,f \in G.$$

$\psi$ is *normalized* if $\psi(g,1) = \psi(1,h) = 1$, for all $g,h \in G$, and we say $M$ is *row/column balanced* if each element of $\langle\zeta_k\rangle$ appears equally often in every non-initial row/column of $M$.

In this case, $\psi$ is *orthogonal*. The set of all cocycles form a group $Z(G, \langle\zeta_k\rangle) \cong Z(G, \mathrm{C}_k)$.

# Outline

- Develop theory of non-existence, i.e., determine the pairs $(n, p)$ such that there are there no cocyclic $\mathrm{BH}(n, p)$s.

# Outline

- Develop theory of non-existence, i.e., determine the pairs $(n, p)$ such that there are there no cocyclic $\mathrm{BH}(n, p)$s.
- Complete a thorough search for cocyclic $\mathrm{BH}(n, p)$ for any pair $(n, p)$ such that $np \leq 100$, not ruled out by the above.

# Outline

- Develop theory of non-existence, i.e., determine the pairs $(n, p)$ such that there are there no cocyclic $\mathrm{BH}(n, p)$s.
- Complete a thorough search for cocyclic $\mathrm{BH}(n, p)$ for any pair $(n, p)$ such that $np \leq 100$, not ruled out by the above.
    - Search for orthogonal cocycles in $Z(G, \mathrm{C}_p)$ for all groups $G$ of order $n$,

# Outline

- Develop theory of non-existence, i.e., determine the pairs $(n, p)$ such that there are there no cocyclic $\mathrm{BH}(n, p)$s.
- Complete a thorough search for cocyclic $\mathrm{BH}(n, p)$ for any pair $(n, p)$ such that $np \leq 100$, not ruled out by the above.
  - Search for orthogonal cocycles in $Z(G, \mathrm{C}_p)$ for all groups $G$ of order $n$,
  - Search for $(n, p, n, n/p)$-central relative difference sets in a central extension $E$ of $\mathrm{C}_p$ by a group $G$ of order $n$.

# Outline

- Develop theory of non-existence, i.e., determine the pairs $(n, p)$ such that there are there no cocyclic $\mathrm{BH}(n, p)$s.
- Complete a thorough search for cocyclic $\mathrm{BH}(n, p)$ for any pair $(n, p)$ such that $np \leq 100$, not ruled out by the above.
  - Search for orthogonal cocycles in $Z(G, \mathrm{C}_p)$ for all groups $G$ of order $n$,
  - Search for $(n, p, n, n/p)$-central relative difference sets in a central extension $E$ of $\mathrm{C}_p$ by a group $G$ of order $n$.
- Organize into equivalence classes.

# Non-existence of cocyclic $\mathrm{BH}(n, p)$s

A $\mathrm{BH}(n, k)$ matrix $M$ is *group developed* over a group $G$ if $M \sim [\phi(gh)]_{g,h \in G}$ for some $\phi : G \to \langle \zeta_k \rangle$.

# Non-existence of cocyclic $\mathrm{BH}(n, p)$s

A $\mathrm{BH}(n, k)$ matrix $M$ is *group developed* over a group $G$ if $M \sim [\phi(gh)]_{g,h \in G}$ for some $\phi : G \to \langle \zeta_k \rangle$.

## Lemma

*Set* $r_j = \mathrm{Re}(\zeta_k^j)$ *and* $s_j = \mathrm{Im}(\zeta_k^j)$. *A group-developed* $\mathrm{BH}(n, k)$ *exists only if there are non-negative integers* $x_0, \ldots, x_{k-1} \in \{0, \ldots, n\}$ *satisfying*

$$\left(\sum_{j=0}^{k-1} r_j x_j\right)^2 + \left(\sum_{j=0}^{k-1} s_j x_j\right)^2 = n$$

*and such that* $\sum_{j=0}^{k-1} x_j = n$.

# Non-existence of cocyclic $\mathrm{BH}(n, p)$s

Let $p$ be an odd prime.

# Non-existence of cocyclic $\mathrm{BH}(n, p)$s

Let $p$ be an odd prime.

## Lemma

*A group-developed* $\mathrm{BH}(n, p)$ *exists only if there are non-negative integers* $x_0, \ldots, x_{p-1} \in \{0, \ldots, n\}$ *satisfying*

- $\sum_{i=0}^{p-1} x_i^2 - x_i = \frac{n}{p}(n-1)$.

# Non-existence of cocyclic $\mathrm{BH}(n, p)$s

Let $p$ be an odd prime.

## Lemma

*A group-developed* $\mathrm{BH}(n, p)$ *exists only if there are non-negative integers* $x_0, \ldots, x_{p-1} \in \{0, \ldots, n\}$ *satisfying*

- $\sum_{i=0}^{p-1} x_i^2 - x_i = \frac{n}{p}(n-1)$.
- $\sum_{i=0}^{p-1} x_i x_{i+j} = \frac{n}{p}(n-1)$, *for all* $1 \leq j \leq p-1$

*where subscripts are read modulo p.*

# Non-existence of cocyclic $\mathrm{BH}(n, p)$s

Let $p$ be an odd prime.

**Lemma**

*A group-developed $\mathrm{BH}(n, p)$ exists only if there are non-negative integers $x_0, \ldots, x_{p-1} \in \{0, \ldots, n\}$ satisfying*

- $\sum_{i=0}^{p-1} x_i^2 - x_i = \frac{n}{p}(n-1)$.
- $\sum_{i=0}^{p-1} x_i x_{i+j} = \frac{n}{p}(n-1)$, *for all* $1 \leq j \leq p-1$

*where subscripts are read modulo p.*

**Theorem**

*If n is p-square-free then a cocyclic $\mathrm{BH}(n, p)$ is equivalent to a group-developed $\mathrm{BH}(n, p)$.*

No cocyclic $\mathrm{BH}(n, p)$s exist for any $(n, p) \in \{(6, 3), (15, 3), (24, 3), (30, 3), (33, 3), (10, 5), (15, 5)\}$.

No cocyclic $\mathrm{BH}(n, p)$s exist for any $(n, p) \in \{(6, 3), (15, 3), (24, 3), (30, 3), (33, 3), (10, 5), (15, 5)\}$.

For all odd primes $p \leq 17$, there is a unique $\mathrm{BH}(p, p)$ up to equivalence. This is the Fourier matrix of order $p$, which is group developed.

No cocyclic $\mathrm{BH}(n, p)$s exist for any $(n, p) \in \{(6, 3), (15, 3), (24, 3), (30, 3), (33, 3), (10, 5), (15, 5)\}$.

For all odd primes $p \leq 17$, there is a unique $\mathrm{BH}(p, p)$ up to equivalence. This is the Fourier matrix of order $p$, which is group developed.

It remains to check for $\mathrm{BH}(n, p)$s for all $(n, p) \in \{(9, 3), (12, 3), (18, 3), (21, 3), (27, 3), (20, 5), (14, 7)\}$.

$|Z(G, \mathrm{C}_p)| \approx p^{|G|-1}$, so a naive search is out of the question.

## The shift action, and orthogonal cocycles in $Z(G, \mathrm{C}_p)$

$|Z(G, \mathrm{C}_p)| \approx p^{|G|-1}$, so a naive search is out of the question.

We define the *shift action* of $G$ on $Z(G, \mathrm{C}_p)$ by $\psi \cdot a = \psi \partial \psi_a$,

# The shift action, and orthogonal cocycles in $Z(G, \mathrm{C}_p)$

$|Z(G, \mathrm{C}_p)| \approx p^{|G|-1}$, so a naive search is out of the question.

We define the *shift action* of $G$ on $Z(G, \mathrm{C}_p)$ by $\psi \cdot a = \psi \partial \psi_a$, where $\psi_a(g) = \psi(a, g)$ and $\partial \psi_a(g, h) = \psi_a(g)^{-1} \psi_a(h)^{-1} \psi_a(gh)$.

# The shift action, and orthogonal cocycles in $Z(G, \mathrm{C}_p)$

$|Z(G, \mathrm{C}_p)| \approx p^{|G|-1}$, so a naive search is out of the question.

We define the *shift action* of $G$ on $Z(G, \mathrm{C}_p)$ by $\psi \cdot a = \psi \partial \psi_a$, where $\psi_a(g) = \psi(a, g)$ and $\partial \psi_a(g, h) = \psi_a(g)^{-1} \psi_a(h)^{-1} \psi_a(gh)$.

The shift action preserves orthogonality, i.e., elements in an orbit under the shift action are either all orthogonal, or none are.

# The shift action, and orthogonal cocycles in $Z(G, \mathrm{C}_p)$

$|Z(G, \mathrm{C}_p)| \approx p^{|G|-1}$, so a naive search is out of the question.

We define the *shift action* of $G$ on $Z(G, \mathrm{C}_p)$ by $\psi \cdot a = \psi \partial \psi_a$, where $\psi_a(g) = \psi(a, g)$ and $\partial \psi_a(g, h) = \psi_a(g)^{-1} \psi_a(h)^{-1} \psi_a(gh)$.

The shift action preserves orthogonality, i.e., elements in an orbit under the shift action are either all orthogonal, or none are.

Let $\Gamma$ be the permutation representation $G \to \mathrm{Sym}(Z(G, \mathrm{C}_p))$ associated to the shift action.

# The shift action, and orthogonal cocycles in $Z(G, \mathrm{C}_p)$

$|Z(G, \mathrm{C}_p)| \approx p^{|G|-1}$, so a naive search is out of the question.

We define the *shift action* of $G$ on $Z(G, \mathrm{C}_p)$ by $\psi \cdot a = \psi \partial \psi_a$, where $\psi_a(g) = \psi(a, g)$ and $\partial \psi_a(g, h) = \psi_a(g)^{-1} \psi_a(h)^{-1} \psi_a(gh)$.

The shift action preserves orthogonality, i.e., elements in an orbit under the shift action are either all orthogonal, or none are.

Let $\Gamma$ be the permutation representation $G \to \mathrm{Sym}(Z(G, \mathrm{C}_p))$ associated to the shift action.

**Theorem**

*Suppose that $|G| = n \geq 5$. Then $\Gamma$ is a faithful representation of $G$ in $\mathrm{GL}(n + r - 1, p)$ where $r$ is the rank of the Sylow $p$-subgroup of $\mathrm{Hom}(H_2(G), C_p)$.*

# The shift action, and orthogonal cocycles in $Z(G, \mathrm{C}_p)$

The matrix group setting enables fast calculation of orbits under the shift action. While the search space is still quite large ($\approx p^{n-1}/n$), it is feasible to calculate the orbits and test representatives for orthogonality for reasonably small $p$ and $n$.

# The shift action, and orthogonal cocycles in $Z(G, \mathrm{C}_p)$

The matrix group setting enables fast calculation of orbits under the shift action. While the search space is still quite large ($\approx p^{n-1}/n$), it is feasible to calculate the orbits and test representatives for orthogonality for reasonably small $p$ and $n$.

The table below gives the number $t$ of orthogonal cocycles found for the given group isotypes, when $p = 3$. Note than none were found for groups of order 18.

| $G$ | $\mathrm{C}_9$ | $\mathrm{C}_3^2$ | $\mathrm{C}_{12}$ | $\mathrm{C}_3 \rtimes \mathrm{C}_4$ | $\mathrm{Alt}(4)$ | $\mathrm{D}_6$ | $\mathrm{C}_2^2 \times \mathrm{C}_3$ |
|---|---|---|---|---|---|---|---|
| $t$ | 18 | 144 | 0 | 288 | 48 | 0 | 96 |

# Central relative difference sets

A *relative* $(v, w, k, \lambda)$*-difference set* in a finite group $E$ of order $vw$ relative to a normal subgroup $N$ of order $w$, is a $k$-subset $R$ of $E$ such that the multiset of quotients $r_i r_j^{-1}$, $(r_i, r_j \in R,\ i \neq j)$, contains each element of $E \setminus N$ exactly $\lambda$ times, and no element of the forbidden subgroup $N$. If $N \leq Z(E)$, then $R$ is *central*.

# Central relative difference sets

A *relative* $(v, w, k, \lambda)$*-difference set* in a finite group $E$ of order $vw$ relative to a normal subgroup $N$ of order $w$, is a $k$-subset $R$ of $E$ such that the multiset of quotients $r_i r_j^{-1}$, $(r_i, r_j \in R,\ i \neq j)$, contains each element of $E \setminus N$ exactly $\lambda$ times, and no element of the forbidden subgroup $N$. If $N \leq Z(E)$, then $R$ is *central*.

Let $E$ be a central extension of $\mathrm{C}_p$ by $G$.

# Central relative difference sets

A *relative* $(v, w, k, \lambda)$*-difference set* in a finite group $E$ of order $vw$ relative to a normal subgroup $N$ of order $w$, is a $k$-subset $R$ of $E$ such that the multiset of quotients $r_i r_j^{-1}$, $(r_i, r_j \in R,\ i \neq j)$, contains each element of $E \setminus N$ exactly $\lambda$ times, and no element of the forbidden subgroup $N$. If $N \leq Z(E)$, then $R$ is *central*.

Let $E$ be a central extension of $\mathrm{C}_p$ by $G$.

**Theorem**

*There exists a cocyclic* $\mathrm{BH}(n, p)$ *if and only if there is a relative* $(n, p, n, n/p)$*-difference set in* $E$ *with central forbidden subgroup* $\mathrm{C}_p$.

# Existence of cocyclic BH($n, p$)s

The table below summarizes existence of matrices in our classification.

| $p \setminus \frac{n}{p}$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | F | N | E | E | N | S2 | S1 | N | E | N | N |
| 5 | F | N | N | S1 | | | | | | | |
| 7 | F | S1 | | | | | | | | | |

N: no cocyclic Butson Hadamard matrices by non-existence theores.
E: cocyclic Butson Hadamard matrices exist.
S1: no cocyclic Butson Hadamard matrices according to an relative difference set search.
S2: no cocyclic Butson Hadamard matrices according to an orthogonal cocycle search.

F: the Fourier matrix is the only Butson Hadamard matrix.

# The classification

There are only 3 interesting cases remaining, all with entries in $\langle \zeta_3 \rangle$.

# The classification

There are only 3 interesting cases remaining, all with entries in $\langle \zeta_3 \rangle$.

- $\mathrm{BH}(9,3)$ : 3 unique equivalence classes.

# The classification

There are only 3 interesting cases remaining, all with entries in $\langle \zeta_3 \rangle$.

- $\mathrm{BH}(9,3)$ : 3 unique equivalence classes.
- $\mathrm{BH}(12,3)$ : 2 unique equivalence classes.

# The classification

There are only 3 interesting cases remaining, all with entries in $\langle \zeta_3 \rangle$.

- $\mathrm{BH}(9,3)$ : 3 unique equivalence classes.
- $\mathrm{BH}(12,3)$ : 2 unique equivalence classes.
- $\mathrm{BH}(27,3)$ : 16 unique equivalence classes.

There are only 3 interesting cases remaining, all with entries in $\langle \zeta_3 \rangle$.

- $\mathrm{BH}(9,3)$ : 3 unique equivalence classes.
- $\mathrm{BH}(12,3)$ : 2 unique equivalence classes.
- $\mathrm{BH}(27,3)$ : 16 unique equivalence classes.

The full classification is available at

`http://www.maths.nuigalway.ie/~dane/BHIndex.html`.

# The classification

There are only 3 interesting cases remaining, all with entries in $\langle \zeta_3 \rangle$.

- $\mathrm{BH}(9,3)$ : 3 unique equivalence classes.
- $\mathrm{BH}(12,3)$ : 2 unique equivalence classes.
- $\mathrm{BH}(27,3)$ : 16 unique equivalence classes.

The full classification is available at

`http://www.maths.nuigalway.ie/~dane/BHIndex.html`.

## Concluding comments

- All matrices found were equivalent to group developed matrices.
- Many of the classes found were previously unknown.
- Various composition techniques may be used to generate higher order Butson matrices from these ones.

# References


- W. Bosma, J. Cannon, and C. Playoust, The Magma algebra system. I. The user language, J. Symbolic Comput., 24, no. 3-4, 235–265,(1997).
- R. Egan, D. L. Flannery, P. Ó Catháin, Classifying cocyclic Butson Hadamard matrices, *Algebraic Design Theory and Hadamard Matrices*, Springer Proc. Math. Stat., 133, in press, 2015.
- D. L. Flannery, R. Egan, On linear shift representations, J. Pure Appl. Algebra, 219, no. 8, 3482–3494, 2015.
- P. Ó Catháin and M. Röder, The cocyclic Hadamard matrices of order less than 40, Des. Codes Cryptogr., 58, no. 1, 73–88, 2011.
- M. Röder, The GAP package RDS, `http://www.gap-system.org/Packages/rds.html`