Combinatorics and Computer Algebra (CoCoA 2015)
Colorado State Univ., Fort Collins, July 19-25, 2015

Combinatorics, Modular Functions, and Computer Algebra

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Numbers

1	1	2	3	5
7	11	15	22	30
42	56	77	101	135
176	231	297	385	490
627	792	1002	1255	1575
1958	2436	3010	3718	4565
5604	6842	8349	10143	12310
14883	17977	21637	26015	31185
37338	44583	53174	63261	75175
89134	105558	124754	147273	173525
204226	239943	281589	329931	386155
451276	526823	614154	715220	831820
966467	1121505	1300156	1505499	1741630
2012558	2323520	2679689	3087735	3554345
4087968	4697205	5392783	6185689	7089500
8118264	9289091	10619863	12132164	13848650

Example: $p(4) = 5$: $4,\ 3+1,\ 2+2,\ 2+1+1,\ 1+1+1+1.$

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EULER. The generating function of the partition numbers is

$$\begin{aligned}
E(q) := \sum_{n=0}^{\infty} p(n) q^n \quad &= \quad \prod_{n=1}^{\infty} \frac{1}{1-q^n} \\
&= \quad (1 + q^1 + q^{1+1} + q^{1+1+1} + \dots) \\
&\times \quad (1 + q^2 + q^{2+2} + q^{2+2+2} + \dots) \\
&\times \quad (1 + q^3 + q^{3+3} + q^{3+3+3} + \dots) \\
&\times \quad \text{etc.} \\
&= \quad \cdots + q^{1+1+1+2+2+3+\cdots} + \cdots
\end{aligned}$$

Back to $p(n)$: ANY PATTERNS?

- 3,313, 3,325, and 3,362 of the first 10,000 entries are congruent respectively to 0, 1, and 2 modulo 3. [Ahlgren & Ono]

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- 3,611 (many more than the expected one-fifth) of the first 10,000 values of $p(n)$ are divisible by 5. [Ahlgren & Ono]

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WHY? Let's look again at our table:

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Ramanujan's famous congruences

$$
\begin{aligned}
p(5n+4) &\equiv 0 \pmod{5},\\
p(7n+5) &\equiv 0 \pmod{7},\\
p(11n+6) &\equiv 0 \pmod{11}
\end{aligned}
$$

Proof. The first congruence is implied by

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\prod_{j=1}^{\infty} \frac{(1-q^{5j})^5}{(1-q^j)^6}$$

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"It would be difficult to find more beautiful formulae than the 'Rogers-Ramanujan' identities . . . ; but here Ramanujan must take second place to Prof. Rogers; and, if I had to select one formula from all Ramanujan's work, I would agree with Major MacMahon in selecting . . ." [G.H. Hardy]

Ramanujan [1919] found also an identity for the 7-case:

$$\sum_{n=0}^{\infty} p(7n+5)q^n$$
$$= 7\prod_{j=1}^{\infty}\frac{(1-q^{7j})^3}{(1-q^j)^4} + 49q\prod_{j=1}^{\infty}\frac{(1-q^{7j})^7}{(1-q^j)^8}.$$

What about the case $p(11n+6)$?

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What about the case $p(11n+6)$?

No such identity has been known, until recently ...

Modular Functions

Recall

$$E(q) := \prod_{n=1}^{\infty}(1 - q^n).$$

We move from formal power (resp. Laurent) series to complex functions by defining

$$q := q(\tau) := \exp(2\pi i\tau)(= e^{2\pi i\tau}),$$

where here and in the following

$$\tau \in \mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}.$$

MODULAR SYMMETRY.

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MODULAR SYMMETRY. If $g(\tau) := q(\tau)\, E\left(q(\tau)\right)^{24}$ then

$$g\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{12} g(\tau) \text{ for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

We also need subgroups where $N \in \mathbb{N} \setminus \{0\}$:

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : N|c \right\}.$$

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Def. A holomorphic function $f(\tau)$ is a modular function for $\Gamma_0(N)$, in short: $f \in M(N)$, if

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = f(\tau) \text{ for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$$

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and for all $p/q \in \mathbb{Q} \cup \{\infty\}$:

$$\lim_{\tau \to p/q} |f(\tau)| \in \mathbb{R}_{\geq 0} \cup \{\infty\}.$$

Examples. Eta quotients like

$$q\,\frac{E(q^5)^6}{E(q)^6} = \prod_{j=1}^{\infty} \frac{(1-q^{5j})^6}{(1-q^j)^6} \in M(5),$$

$$q\,\frac{E(q^7)^4}{E(q)^4} = \prod_{j=1}^{\infty} \frac{(1-q^{7j})^4}{(1-q^j)^4} \in M(7);$$

functions like

$$q\,E(q^5)\sum_{n=0}^{\infty} p(5n+4)q^n \in M(5),$$

or the celebrated modular j-function (Klein invariant),

$$j(\tau) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + 864299970q^3 + \cdots \in M(1).$$

NOTE. $\Gamma_0(1) = \mathrm{SL}_2(\mathbb{Z})$.

Zero recognition of functions in $M(N)$?

The action

$$\Gamma_0(N) \times \mathbb{H} \to \mathbb{H}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau := \frac{a\tau + b}{c\tau + d}$$

induces the action

$$\Gamma_0(N) \times \mathrm{HoloFus}(\mathbb{H}) \to \mathrm{HoloFus}(\mathbb{H}), (f|\gamma)(\tau) := f(\gamma\tau).$$

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$$X_0(N) := \{[\tau] : \tau \in \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}\} \quad \text{(set of orbits)}.$$

The set of orbits

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Each (holomorphic) $f \in M(N)$ has a meromorphic extension

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In our context we can restrict to "good" subalgebras of $M(N)$:

To prove (and to find!) Ramanujan's identities automatically, we can restrict to consider

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Our goal is to present a given $f \in M^{\infty}(N)$ in the form

$$f = c_1 \frac{E(q^{a_1})^{\alpha_1} E(q^{a_2})^{\alpha_2} \cdots}{E(q^{b_1})^{\beta_1} E(q^{b_2})^{\beta_2} \cdots} + \cdots + c_m \frac{E(q^{u_1})^{\mu_1} E(q^{u_2})^{\mu_2} \cdots}{E(q^{v_1})^{\nu_1} E(q^{v_2})^{\nu_2} \cdots}$$

with coefficients $c_j \in \mathbb{C}$ and the Eta quotients in $M^{\infty}(N)$.

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with coefficients $c_j \in \mathbb{C}$ and the Eta quotients in $M^{\infty}(N)$.

All Eta quotients in $M^{\infty}(N)$ form a multiplicative monoid denoted by

$$E^{\infty}(N) := \{f \in M^{\infty}(N) : f \text{ is an Eta quotient}\}$$

NOTE 1. This monoid is finitely generated,

$$E^{\infty}(N) = \langle f_1, \ldots, f_n \rangle = \{f_1^{\ell_1} \ldots f_n^{\ell_n} : \ell_i \in \mathbb{N}\};$$

the generators $f_1, \ldots, f_n \in E^{\infty}(N)$ for fixed N can be determined algorithmically (e.g., with partition analysis).

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NOTE 2. In view of

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our task is to constructively decide subalgebra membership

$$f \overset{?}{\in} \mathbb{C}[f_1, \dots, f_n] \subseteq M^{\infty}(N).$$

Monoids & Subalgebras

McNuggets Partitions

GIVEN

TASK: buy exactly 43 nuggets.

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IMPOSSIBLE!

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See: "How to order 43 Chicken McNuggets - Numberphile"
www.youtube.com/watch?v=vNTSugyS038

Generators of (additive) submonoids of $\mathbb{N}$

M:=$\langle 6, 9, 20\rangle \subseteq \mathbb{N} = \{0, 1, \ldots\}$

$$[M]_i := \{x \in M : x \equiv i \pmod{6}\}$$

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$[M]_0 = \{0, 6, 12, \dots\}$,
$[M]_1 = \{\not{1}, \not{7}, \not{13}, \not{19}, \not{25}, \not{31}, \not{37}, \not{43}, 49, 54, \dots\}$,
$[M]_2 = \{\not{2}, \not{8}, \not{14}, 20, 26, \dots\}$,
$[M]_3 = \{\not{3}, 9, 15, \dots\}$,
$[M]_4 = \{\not{4}, \not{10}, \not{16}, \not{22}, \not{28}, \not{34}, 40, 46, \dots\}$,
$[M]_5 = \{\not{5}, \not{11}, \not{17}, \not{23}, 29, 35, \dots\}$.

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$M = \langle 6, 9, 20\rangle = \langle 6, 49, 20, 9, 40, 29\rangle$ and

$$\{u_1, u_2, u_3, u_4, u_5\} := \{49, 20, 9, 40, 29\} \equiv \{1, 2, 3, 4, 5\} \pmod{6}.$$

QUESTION. What happens if we add more generators?

$M^+ := \langle 4, 6, 9, 20 \rangle \subseteq \mathbb{N} = \{0, 1, \dots\}$

$$[M^+]_i := \{x \in M : x \equiv i \pmod{6}\}$$

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$$\begin{aligned}
[M^+]_0 &= \{0, 6, 12, \ldots\}, \\
[M^+]_1 &= \{\cancel{1}, \cancel{7}, 13, 19, \ldots\}, \\
[M^+]_2 &= \{\cancel{2}, 8, 14, \ldots\}, \\
[M^+]_3 &= \{\cancel{3}, 9, 15, \ldots\}, \\
[M^+]_4 &= \{4, 10, \ldots\}, \\
[M^+]_5 &= \{\cancel{5}, \cancel{11}, 17, 23, \ldots\}.
\end{aligned}$$

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[M^+]_4 &= \{4, 10, \dots\},\\
[M^+]_5 &= \{\cancel{5}, \cancel{11}, 17, 23, \dots\}.
\end{aligned}$$

NOTE. In contrast to the Frobenius number $F(6, 9, 20) = 43$, now $F(4, 6, 9, 20) = 11$ and

$$\begin{aligned}
43 &= 2\,④ + ⑥ + ⑨ + 1\,⑳\\
&= ④ + 2\,⑥ + 3\,⑨ = \text{[four more]}.
\end{aligned}$$

QUESTION. How to find presentations in terms of generators?

(a) Greedy Algorithm

STRATEGY: Successive subtraction; begin with the largest generator, first taken a maximal number of times; iterate.

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EXAMPLE: $x = 62$ and $M = \langle \textcircled{6}, \textcircled{9}, \textcircled{20} \rangle$.

$62 - 3\textcircled{20} = 2$ ✗

$62 - 2\textcircled{20} = 22,$

$22 - 2\textcircled{9} = 4$ ✗

$22 - 1\textcircled{9} = 13,$

$13 - 2\textcircled{6} = 1$ ✗

$22 - 3\textcircled{6} = 4$ ✗

$62 - 1\textcircled{20} = 42,$

$42 - 4\textcircled{9} = 6,$

$6 - 1\textcircled{6} = 0$ ✓.

SOLUTION. $62 = 1 \, \textcircled{6} + 4 \, \textcircled{9} + 1 \, \textcircled{20}$.

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Also,

$$62 = 4 \textcircled{6} + 2 \textcircled{9} + 1 \textcircled{20}$$

and

$$62 = 7 \textcircled{6} + 0 \textcircled{9} + 1 \textcircled{20}.$$

SOLUTION. $62 = 1 \enclose{circle}{6} + 4 \enclose{circle}{9} + 1 \enclose{circle}{20}$.

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and

$$62 = 7 \enclose{circle}{6} + 0 \enclose{circle}{9} + 1 \enclose{circle}{20}.$$

PREVIEW. The Omega package computes

In[2]:=OEqR[

OEqSum[$x^a y^b z^c, \{6a + 9b + 20c = 62\}, \lambda$]]

Out[2]= $\left(x^7 + x^4 y^2 + x y^4\right) z$.

(b) Algorithm based on the generators $u_1, \ldots, u_5$

STRATEGY: Find a representation $x = u_i + \ell \textcircled{6}$.

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EXAMPLE: $M = \langle 6, u_1, u_2, u_3, u_4, u_5 \rangle = \langle 6, 43, 20, 9, 40, 29 \rangle$.
and $x = 62$.

(b) Algorithm based on the generators $u_1, \dots, u_5$

STRATEGY: Find a representation $x = u_i + \ell \textcircled{6}$.

EXAMPLE: $M = \langle 6, u_1, u_2, u_3, u_4, u_5 \rangle = \langle 6, 43, 20, 9, 40, 29 \rangle$.
and $x = 62$.

$62 \equiv 2 \pmod 6$; hence we recall that

$$[M]_2 = \{\not{2}, \not{8}, \not{14}, 20, 26, \dots\};$$

consequently,

$$62 = 20 + 7\textcircled{6} = u_2 + 7\textcircled{6}.$$

Application: $\mathbb{C}$-subalgebras of $\mathbb{C}[z]$

GIVEN: $f_1, f_2, f_3 \in \mathbb{C}[z]$ with

$$\begin{aligned} t := f_1 &= z^6 + a\,z^5 + \dots, \\ f_2 &= z^9 + b\,z^8 + \dots, \\ f_3 &= z^{20} + c\,z^{19} + \dots; \end{aligned}$$

FIND: $g_1, \dots, g_k \in \mathbb{C}[z]$ such that

$$\begin{aligned} \mathbb{C}[f_1, f_2, f_3] &= \langle g_1, \dots, g_k \rangle_{\mathbb{C}[t]} \\ &= \mathbb{C}[t] + \mathbb{C}[t]\,g_1 + \dots + \mathbb{C}[t]\,g_k. \end{aligned}$$

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KEY IDEA:

$$z^n + \dots = z^{u_i + \ell \textcircled{6}} + \dots = (z^{a_i \textcircled{9} + b_i \textcircled{20}} + \dots)\,t^{\ell} + \dots.$$

DECISION PROCEDURE: $\mathbb{C}$-subalgebra membership

GIVEN: f and $f_1, \ldots, f_n$ from $\mathbb{C}[z]$;

FIND: polynomials $p_0, p_1, \ldots, p_k \in \mathbb{C}[z]$ such that

$$\begin{aligned} f &= p_0(t) + p_1(t)\, g_1 + \cdots + p_k(t)\, g_k \\ &\in \langle 1, g_1, \ldots, g_k \rangle_{\mathbb{C}[t]} = \mathbb{C}[f_1, \ldots, f_n] \end{aligned}$$

APPLICATION.

DECISION PROCEDURE: $\mathbb{C}$-subalgebra membership

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APPLICATION. With this procedure the right hand sides of Ramanujan identities can be found automatically: just apply it to given $f \in M^{\infty}(N)$ and $f_1, \ldots, f_n \in E^{\infty}(N)$. In this context, $z = \frac{1}{q}$ because of possible poles sitting at $[\infty]$; the f_j can be treated as formal Laurent series like

$$\begin{aligned} f_1 &= \frac{a_6}{q^6} + \frac{a_5}{q^5} + \cdots, \\ f_2 &= \frac{b_9}{q^9} + \frac{b_8}{q^8} + \cdots, \\ f_3 &= \frac{c_{20}}{q^{20}} + \frac{c_{19}}{q^{19}} + \cdots, \text{ etc.} \end{aligned}$$

Example. Radu's "Ramanujan-Kolberg" package in STEP 1 computes that:

$$q^{-1}\frac{E(q)^8}{E(q^7)^7}\sum_{n=0}^{\infty}p(7n+5)q^n \in M^{\infty}(7).$$

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In STEP 4 the package finds Ramanujan's identity:

$$q^{-1}\frac{E(q)^8}{E(q^7)^7}\sum_{n=0}^{\infty}p(7n+5)q^n = 49\cdot 1 + 7\cdot f_1.$$

QUESTION. How to compute the $u_1, \ldots, u_5$ from $M = \langle 6, 9, 20 \rangle$?

ANSWER. To find $u_1 = 49$, we ask Omega to compute

In[3]:=OEqR[
OEqSum[$x^a y^b z^c, \{-6a + 9b + 20c = 1\}, \lambda$]]

Out[3]= $\dfrac{x^8 y z^2}{\left(1-x^3 y^2\right)\left(1-x^{10} z^3\right)}$.

`Out[3]` means that

$$\sum_{\substack{a,b,c\geq 0 \\ s.t.\ -6a+9b+20c=1}} x^a y^b z^c = \frac{x^8 y z^2}{(1-x^3y^2)\,(1-x^{10}z^3)};$$

in other words,

$$\begin{pmatrix}8\\1\\2\end{pmatrix} + \mathbb{N}\begin{pmatrix}3\\2\\0\end{pmatrix} + \mathbb{N}\begin{pmatrix}10\\0\\3\end{pmatrix}$$

is the solution set of $-6a+9b+20c=1$. For example,

$$1\,\textcircled{9} + 2\,\textcircled{20} = 1 + 8\,\textcircled{6} = 49 = u_1.$$

Analogously one computes $u_2, \ldots, u_5$.

Partition Analysis

PROBLEM [Polya]. Consider triangles with sides of integer length such that

$$1 \leq \ell \leq m \leq n;$$

TASK: express the total number of such triangles in terms of n.

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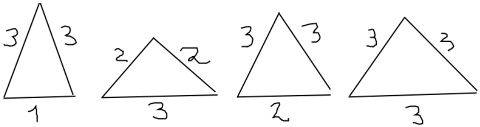
EXAMPLE. $n = 3$



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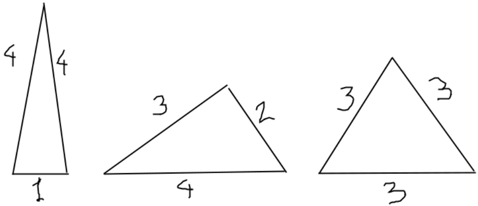

SOLUTION IDEA. Determine the generating function

$$T(x,y,z;q) := \sum_{\substack{c \geq b \geq a \geq 1, \\ \text{s.t. } a+b>c}} x^a y^b z^c q^{a+b+c}.$$

Then all Polya triangles are given by the

coefficient of z^n in $T(x,y,z;1)$,

and the total number of such triangles as the

coefficient of z^n in $T(1,1,z;1)$.

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$$T(x,y,z;q) := \sum_{\substack{c\geq b\geq a\geq 1, \\ \text{s.t. } a+b>c}} x^a y^b z^c q^{a+b+c}.$$

Then all triangles with given perimeter n are given by the

coefficient of q^n in $T(x,y,z;q)$,

and the total number of such triangles as the

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HOW TO FIND A SUITABLE REPRESENTATION OF

$$T(x,y,z;q) := \sum_{\substack{c\geq b\geq a\geq 1,\\ \text{s.t. } a+b>c}} x^a y^b z^c q^{a+b+c}?$$

Pick up the Omega package, freely available at
`www.risc.jku.at/research/combinat/software`
and do the following:

```
In[1]:= << RISC`Omega`
```

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Omega Package by Axel Riese (in cooperation with George E. Andrews and Peter Paule) - ©RISC, JKU Linz - V 2.47

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```
In[2]:= OR[OSum[x^a y^b z^c q^(a+b+c),
          {a ≥ 1, b ≥ a, c ≥ b, a + b > c}, 1]]
```

```
In[2]:= OR[OSum[x^a y^b z^c q^(a+b+c),
           {a ≥ 1, b ≥ a, c ≥ b, a + b > c}, l]]

        Assuming b ≥ 0

        Assuming c ≥ 0

        Eliminating l2...

        Eliminating l3...

        Eliminating l1...
```

Out[2]=

$$\frac{q^3\,x\,y\,z}{\left(1-q^2\,y\,z\right)\,\left(1-q^3\,x\,y\,z\right)\,\left(1-q^4\,x\,y\,z^2\right)}$$

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WHAT STANDS BEHIND THIS?

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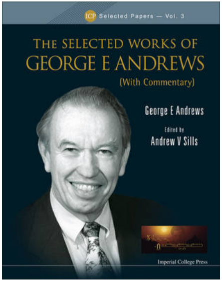
WHAT STANDS BEHIND THIS? MacMahon's Partition Analysis

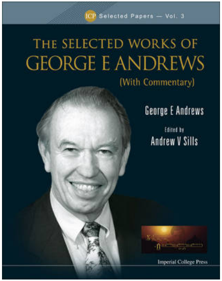


"The no. of partitions of N of the form $N = b_1 + \cdots + b_n$ satisfying

$$\frac{b_n}{n} \geq \frac{b_{n-1}}{n-1} \geq \cdots \geq \frac{b_2}{2} \geq \frac{b_1}{1} \geq 0$$

equals the no. of partitions of N into odd parts each $\leq 2n-1$.

This problem cried out for MacMahon's Partition Analysis, . . .



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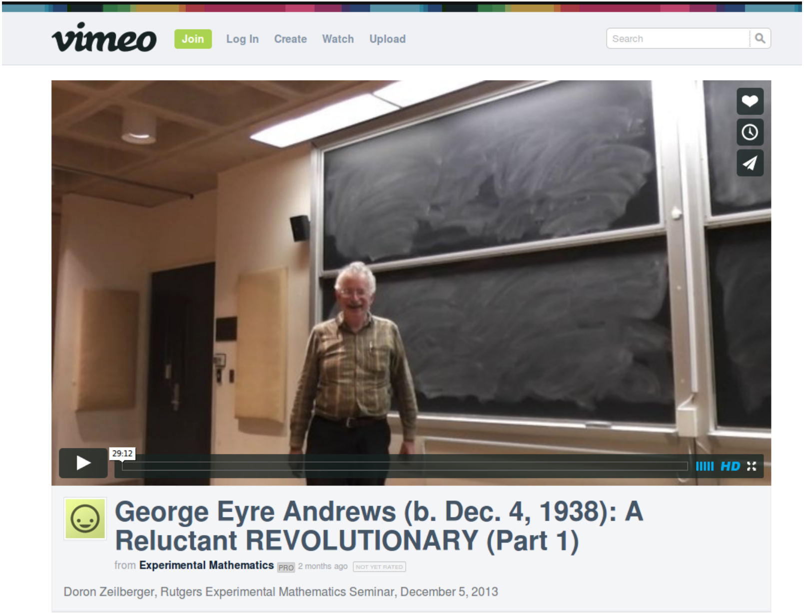
equals the no. of partitions of N into odd parts each $\leq 2n-1$.

This problem cried out for MacMahon's Partition Analysis, ...

Given that Partition Analysis is an algorithm for producing partition generating functions, I was able to convince Peter Paule and Axel Riese to join an effort to automate this algorithm."

How Zeilberger tells the story of partition analysis (and more):



MacMahon's Omega Calculus:

$$\begin{aligned} T(x,y,z;q) &:= \sum_{\substack{c\geq b\geq a\geq 1, \\ \text{s.t. } a+b>c}} x^a y^b z^c q^{a+b+c} \\ &= \underset{\geq}{\Omega} \sum_{a,b,c\geq 1} l_1^{b-a} l_2^{c-b} l_3^{a+b-1-c} x^a y^b z^c q^{a+b+c} \end{aligned}$$

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In[3]:=
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OSum[x^a y^b z^c q^(a+b+c),
  {a ≥ 1, b ≥ a, c ≥ b, a + b - 1 - c ≥ 0}, l]
```

Assuming b ≥ 0

Assuming c ≥ 0

Out[3]= $$\underset{l_1, l_2, l_3}{\underset{\geq}{\Omega}} \frac{q\,x}{l_1 \left(1 - \frac{q\,z\,l_2}{l_3}\right) \left(1 - \frac{q\,x\,l_3}{l_1}\right) \left(1 - \frac{q\,y\,l_1\,l_3}{l_2}\right)}$$

The second step is the elimination of the slack variables:

In[4]:= **OR[%]**

```
Eliminating l2...
Eliminating l3...
Eliminating l1...
```

Out[4]= $$\frac{q^3\, x\, y\, z}{(1 - q^2\, y\, z)\,(1 - q^3\, x\, y\, z)\,(1 - q^4\, x\, y\, z^2)}$$

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NOTE. To this end, MacMahon used elimination rules like

$$\underset{\geqq}{\Omega}\, \frac{l^{-k}}{(1-A\,l)\left(1-\frac{B}{l}\right)} = \frac{A^k}{(1-A)(1-A\,B)}, \quad k \geq 0.$$

GENERAL THEME: linear Diophantine constraints

- Find $b_1, \ldots b_n \in \mathbb{N}$ such that

$$\begin{pmatrix} c_{1,1} & \cdots & c_{1,n} \\ c_{2,1} & \cdots & c_{2,n} \\ \vdots & \ddots & \vdots \\ c_{m,1} & \cdots & c_{m,n} \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \geq \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix}$$

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- New algorithm by F. Breuer & Z. Zafeirakopolous

  [arXiv:1501.07773] The new algorithm Polyhedral Omega solves linear Diophantine systems by combining methods from partition analysis with methods from polyhedral geometry as Brion decompositions and Barvinok's short rational function representations.

Omega Discovery

MacMahon's plane partitions, e.g., of $n = 3$:

$$3,\ 2+1,\ \begin{array}{c}2\\+\\1\end{array},\ 1+1+1,\ \begin{array}{c}1\\+\\1\\+\\1\end{array},\ \begin{array}{cc}1+1\\+\\1\end{array}.$$

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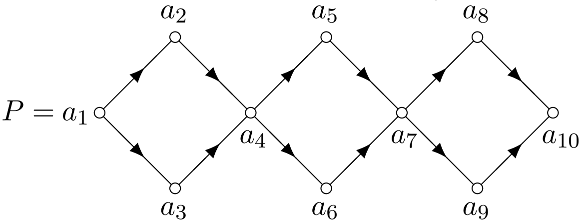
More generally, one can consider digraphs (resp. posets) like



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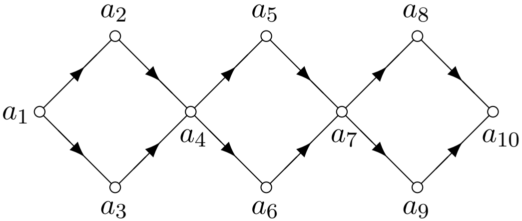
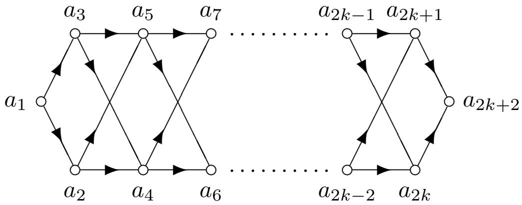
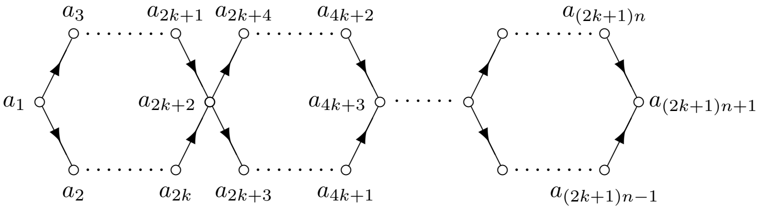
$P =$ 

together with the associated generating functions; here:

$$f(q) := \sum_{a_1,\dots,a_{10}\in\mathbb{N} \text{ s.t. } P} q^{a_1+\cdots+a_{10}}.$$



A k-elongated partition diamond of length 1



A k-elongated partition diamond of length n

Generating function for k-elongated diamonds of length n:

$$h_{n,k}(q) = \frac{\prod_{j=0}^{n-1}(1+q^{(2k+1)j+2})(1+q^{(2k+1)j+4})\cdots(1+q^{(2k+1)j+2k})}{\prod_{j=1}^{(2k+1)n+1}(1-q^j)}$$

Andrews' great idea: delete the source:

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and glue the diamonds together:

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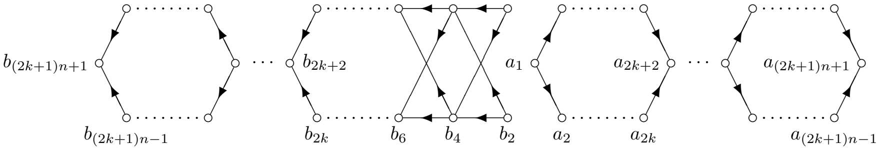

A broken k-diamond of length $2n$

Consequently,

$$\sum_{m=0}^{\infty} \Delta_k(m) q^m := \lim_{n\to\infty} h_{n,k}(q)\, h^*_{n,k}(q)$$

$$= \prod_{j=1}^{\infty} \frac{(1-q^{2j})(1-q^{(2k+1)j})}{(1-q^j)^3(1-q^{(4k+2)j})}$$

$$= \frac{E(q^2)E(q^{2k+1})}{E(q)^3E(q^{4k+2})}$$

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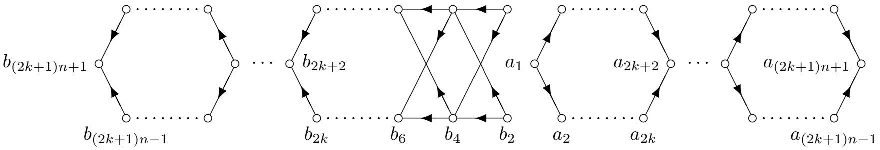
— and think of Ramanujan!



A broken k-diamond of length $2n$

In[1]:= $\mathrm{bd}[n\_, k\_] := \prod_{j=1}^{n} \frac{(1 - q^{2 j})\,(1 - q^{(2 k+1) j})}{(1 - q^{j})^{3}\,(1 - q^{(4 k+2) j})}$

```
In[6]:= bd1 = Normal[Series[bd[30, 1], {q, 0, 30}]]
```

$$1 + 3 q + 8 q^2 + 18 q^3 + 38 q^4 + 75 q^5 + 142 q^6 + 258 q^7 + 455 q^8 + 780 q^9 + 1308 q^{10} + 2148 q^{11} + 3467 q^{12} + 5505 q^{13} + 8618 q^{14} + 13\,314 q^{15} + 20\,327 q^{16} + 30\,693 q^{17} + 45\,882 q^{18} + 67\,944 q^{19} + 99\,745 q^{20} + 145\,239 q^{21} + 209\,882 q^{22} + 301\,128 q^{23} + 429\,148 q^{24} + 607\,710 q^{25} + 855\,414 q^{26} + 1\,197\,228 q^{27} + 1\,666\,585 q^{28} + 2\,308\,014 q^{29} + 3\,180\,668 q^{30}$$

```
In[7]:= Mod[CoefficientList[bd1, q], 2]

{1, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 1, 0, 0, 1, 1, 0, 0, 1, 1,
 0, 0, 0, 0, 0, 0, 1, 0, 0}

In[8]:= Mod[CoefficientList[bd1, q], 3]

{1, 0, 2, 0, 2, 0, 1, 0, 2, 0, 0, 0, 2, 0, 2, 0, 2, 0, 0, 0, 1, 0,
 2, 0, 1, 0, 0, 0, 1, 0, 2}

In[9]:= Mod[CoefficientList[bd1, q], 4]

{1, 3, 0, 2, 2, 3, 2, 2, 3, 0, 0, 0, 3, 1, 2, 2, 3, 1, 2, 0, 1, 3,
```

Theorem. For all $n \in \mathbb{N}$,

$$\Delta_1(2n+1) \equiv 0 \pmod{3}.$$

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Proof. Because of $(1-q^j)^3 \equiv 1-q^{3j} \pmod 3$,

$$\sum_{m=0}^{\infty} \Delta_1(m) q^m = \prod_{j=1}^{\infty} \frac{(1-q^{2j})(1-q^{3j})}{(1-q^j)^3(1-q^{6j})}$$

$$\equiv \prod_{j=1}^{\infty} \frac{(1-q^{2j})(1-q^{3j})}{(1-q^{3j})(1-q^{6j})} \pmod 3.$$

Hence the coefficients of odd powers of q have to be zero.

Recall:

Theorem. For all $n \in \mathbb{N}$,

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Algorithmic Proof [Radu 2014]:

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$$\sum_{n=0}^{\infty} \Delta_1(2n+1) q^n = 3 \prod_{j=1}^{\infty} \frac{(1-q^{2j})^2(1-q^{6j})^2}{(1-q^j)^6}$$

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NOTE. Human proof [Hirschhorn & Sellers, 2007]

NOTE. There is much more: many new families of combinatorial objects, q-series identities, and computer proofs and findings.

Let us return to Ramanujan's congruence

$$p(11n+6) \equiv 0 \pmod{11}:$$

Radu's "Ramanujan-Kolberg" package computes in $M^{\infty}(22)$:

$$\frac{E(q)^{10}E(q^2)^2E(q^{11})^{11}}{E(q^{22})^{22}}\sum_{n=0}^{\infty}p(11n+6)q^n$$

$$\begin{aligned} = \quad & q^{14}\big(1078t^4+13893t^3+31647t^2+11209t-21967 \\ & +z_1(187t^3+5390t^2+594t-9581) \\ & +z_2(11t^3+2761t^2+5368t-6754)\big) \end{aligned}$$

with

Radu's "Ramanujan-Kolberg" package computes in $M^{\infty}(22)$:

$$\frac{E(q)^{10}E(q^2)^2E(q^{11})^{11}}{E(q^{22})^{22}}\sum_{n=0}^{\infty}p(11n+6)q^n$$
$$= \quad q^{14}\big(1078t^4+13893t^3+31647t^2+11209t-21967$$
$$+z_1(187t^3+5390t^2+594t-9581)$$
$$+z_2(11t^3+2761t^2+5368t-6754)\big)$$

with

$$t:=\frac{3}{88}w_1+\frac{1}{11}w_2-\frac{1}{8}w_3,\ z_1:=-\frac{5}{88}w_1+\frac{2}{11}w_2-\frac{1}{8}w_3-3,$$
$$z_2:=\frac{1}{44}w_1-\frac{3}{11}w_2+\frac{5}{4}w_3,$$

where

Radu's "Ramanujan-Kolberg" package computes in $M^{\infty}(22)$:

$$\frac{E(q)^{10}E(q^2)^2E(q^{11})^{11}}{E(q^{22})^{22}}\sum_{n=0}^{\infty}p(11n+6)q^n$$
$$= q^{14}\big(1078t^4+13893t^3+31647t^2+11209t-21967$$
$$+z_1(187t^3+5390t^2+594t-9581)$$
$$+z_2(11t^3+2761t^2+5368t-6754)\big)$$

with

$$t:=\frac{3}{88}w_1+\frac{1}{11}w_2-\frac{1}{8}w_3,\; z_1:=-\frac{5}{88}w_1+\frac{2}{11}w_2-\frac{1}{8}w_3-3,$$
$$z_2:=\frac{1}{44}w_1-\frac{3}{11}w_2+\frac{5}{4}w_3,$$

where

$$w_1:=[-3,3,-7],\, w_2:=[8,4,-8],\, w_3:=[1,11,-11]\in E^{\infty}(22)$$

and

Radu's "Ramanujan-Kolberg" package computes in $M^\infty(22)$:

$$\frac{E(q)^{10}E(q^2)^2E(q^{11})^{11}}{E(q^{22})^{22}}\sum_{n=0}^{\infty}p(11n+6)q^n$$
$$= q^{14}\big(1078t^4+13893t^3+31647t^2+11209t-21967$$
$$+z_1(187t^3+5390t^2+594t-9581)$$
$$+z_2(11t^3+2761t^2+5368t-6754)\big)$$

with

$$t:=\frac{3}{88}w_1+\frac{1}{11}w_2-\frac{1}{8}w_3,\ z_1:=-\frac{5}{88}w_1+\frac{2}{11}w_2-\frac{1}{8}w_3-3,$$
$$z_2:=\frac{1}{44}w_1-\frac{3}{11}w_2+\frac{5}{4}w_3,$$

where

$$w_1:=[-3,3,-7],w_2:=[8,4,-8],w_3:=[1,11,-11]\in E^\infty(22)$$

and

$$[a,b,c]:=q^{-5}\frac{E(q^2)^aE(q^{11})^bE(q^{22})^c}{E(q)^{a+b+c}}.$$

References

- G.E. Andrews and P. Paule: MacMahon's Partition Analysis XI: Broken diamonds and modular forms. Acta Arithmetica 126 (2007), 281-294.
- S. Radu: An Algorithmic Approach to Ramanujan-Kolberg Identities, Journal for Symbolic Computation 68 (2014), 1-33.
- P. Paule and S. Radu: Partition Analysis, Modular Functions, and Computer Algebra. To appear in Recent Trends in Combinatorics, IMA Volume in Mathematics and its Applications, Springer, 2015. (Available at `www.dk-compmath.jku.at/publications`.)

RISC Software & Appls.

Software of the RISC Algorithmic Combinatorics Group

Feel free to pick up packages!

`www.risc.jku.at/research/combinat/software`

Symbolic Summation

Hypergeometric Summation

- fastZeil, the Paule/Schorn implementation of Gosper's and Zeilberger's algorithm in Mathematica (b
- Zeilberger, a Maxima implementation of Gosper's and Zeilberger's algorithm (by F. Caruso).
- MultiSum, a Mathematica package for proving hypergeometric multi-sum identities (by K. Wegscha

q-Hypergeometric Summation

- qZeil, a Mathematica implementation of q-analogues of Gosper's and Zeilberger's algorithm (by A. F
- Bibasic Telescope (pqTelescope), a Mathematica implementation of a generalization of Gosper's alg
- qMultiSum, a Mathematica package for proving q-hypergeometric multi-sum identities (by A. Riese)

Multi-Summation in Difference Fields

- Sigma, a Mathematica package for discovering and proving multi-sum identities (by C. Schneider).

Symbolic Summation for Stirling Numbers

- Stirling, a Mathematica package for computing recurrence equations of sums involving Stirling num

Symbolic Summation and Integration for Holonomic Functions

- HolonomicFunctions, a Mathematica package for dealing with multivariate holonomic functions, incl

Sequences and Power Series

- Asymptotics, a Mathematica package for computing asymptotic series expansions of univariate holonom
- Dependencies, a Mathematica package for computing algebraic relations of C-finite sequences and multi
- Engel, a Mathematica implementation of q-Engel Expansion (by B. Zimmermann).
- GeneratingFunctions, a Mathematica package for manipulations of univariate holonomic functions and se
- ore_algebra, a Sage package for doing computations with Ore operators (by M. Kauers, M. Jaroschek, F
- qGeneratingFunctions, a Mathematica package for manipulations of univariate q-holonomic functions and
- Guess, a Mathematica package for guessing multivariate recurrence equations (by M. Kauers).
- RLangGFun, a Maple implementation of the inverse Schützenberger methodology (by C. Koutschan).

Special Function Algorithms for Indefinite Nested Sums and Integrals

- HarmonicSums, a Mathematica package for dealing with harmonic sums, generalized harmonic sums an

Permutation Groups

- PermGroup, a Mathematica package for permutation groups, group actions and Polya theory (by T. Bayer

Partition Analysis

- Omega, a Mathematica implementation of Partition Analysis (by A. Riese).
- GenOmega, a Mathematica implementation of Guo-Niu Han's general Algorithm for MacMahon's Partition Ana

Difference/Differential Equations

- DiffTools, a Mathematica implementation of several algorithms for solving linear difference equations with poly
- OreSys, a Mathematica implementation of several algorithms for uncoupling systems of linear Ore operator eq
- RatDiff, a Mathematica implementation of Mark van Hoeij's algorithm for finding rational solutions of linear diffe
- SumCracker, a Mathematica implementation of several algorithms for identities and inequalities of special seq

Misc

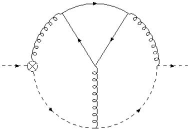
- Singular, a Mathematica interface to the Singular system (by M. Kauers and V. Levandovskyy).
- ModularGroup, a Mathematica package providing basic algorithms and visualization routines related to the T. Ponweiser).

Symbolic Summation in QFT

JKU Collaboration with DESY (Berlin–Zeuthen)
(Deutsches Elektronen–Synchrotron)

Project leader:	Carsten Schneider (RISC)
Partners:	Johannes Blümlein (DESY)
	Peter Paule (RISC)

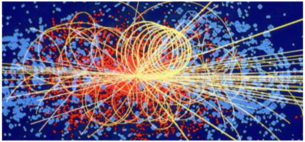
Evaluation of Feynman diagrams



Behavior of particles

$$\int \Phi(N,\epsilon,x)dx$$

Feynman integrals



Evaluations required for the LHC experiment at CERN

DESY

$$\sum f(N,\epsilon,k)$$

multi-sums

RISC

(symbolic summation)

simple sum expressions

processable by physicists

Challenges of the project

About 1000 difficult Feynman diagrams have been treated so far

(some took 50 days of calculation time)

↓

About a million multi-sums have been simplified

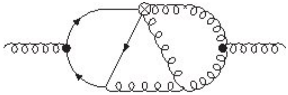
(most were double and triple sums)

Resources

- 9 full time employed researchers at RISC/DESY:

  J. Ablinger, A. Behring, J. Blümlein, A. Hasselhuhn, A. de Freitas, C. Raab, M. Round, C. Schneider, F. Wissbrock

- 4 up-to-date mainframe DESY computers at RISC
  + exploiting DESY's computer farms
- New computer algebra/special functions technologies
  (new/tuned algorithms, efficient implementations,...)



$$= F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + \boxed{F_0(N)}$$

Simplify

$$||$$

$$\sum_{j=0}^{N-3}\sum_{k=0}^{j}\sum_{l=0}^{k}\sum_{q=0}^{-j+N-3}\sum_{s=1}^{-l+N-q-3}\sum_{r=0}^{-l+N-q-s-3}(-1)^{-j+k-l+N-q-3}\times$$

$$\times\frac{\binom{j+1}{k+1}\binom{k}{l}\binom{N-1}{j+2}\binom{-j+N-3}{q}\binom{-l+N-q-3}{s}\binom{-l+N-q-s-3}{r}r!(-l+N-q-r-s-3)!(s-1)}{(-l+N-q-2)!(-j+N-1)(N-q-r-s-2)(q+s+1)}$$

$$\Big[4S_1(-j+N-1)-4S_1(-j+N-2)-2S_1(k)$$

$$-(S_1(-l+N-q-2)+S_1(-l+N-q-r-s-3)-2S_1(r+s))$$

$$+2S_1(s-1)-2S_1(r+s)\Big] + \textbf{3 further 6–fold sums}$$

$\boxed{F_0(N)} =$ (using `Sigma.m`, `EvaluateMultiSums.m` and J. Ablinger's `HarmonicSums.m` package)

$$\frac{7}{12}S_1(N)^4 + \frac{(17N+5)S_1(N)^3}{3N(N+1)} + \Big(\frac{35N^2-2N-5}{2N^2(N+1)^2} + \frac{13S_2(N)}{2} + \frac{5(-1)^N}{2N^2}\Big)S_1(N)^2$$
$$+ \Big(-\frac{4(13N+5)}{N^2(N+1)^2} + \Big(\frac{4(-1)^N(2N+1)}{N(N+1)} - \frac{13}{N}\Big)S_2(N) + \Big(\frac{29}{3} - (-1)^N\Big)S_3(N)$$
$$+ \big(2+2(-1)^N\big)S_{2,1}(N) - 28S_{-2,1}(N) + \frac{20(-1)^N}{N^2(N+1)}\Big)S_1(N) + \Big(\frac{3}{4} + (-1)^N\Big)S_2(N)^2$$
$$- 2(-1)^N S_{-2}(N)^2 + S_{-3}(N)\Big(\frac{2(3N-5)}{N(N+1)} + \big(26+4(-1)^N\big)S_1(N) + \frac{4(-1)^N}{N+1}\Big)$$
$$+ \Big(\frac{(-1)^N(5-3N)}{2N^2(N+1)} - \frac{5}{2N^2}\Big)S_2(N) + S_{-2}(N)\big(10S_1(N)^2 + \Big(\frac{8(-1)^N(2N+1)}{N(N+1)}$$
$$+ \frac{4(3N-1)}{N(N+1)}\Big)S_1(N) + \frac{8(-1)^N(3N+1)}{N(N+1)^2} + \big(-22+6(-1)^N\big)S_2(N) - \frac{16}{N(N+1)}\Big)$$
$$+ \Big(\frac{(-1)^N(9N+5)}{N(N+1)} - \frac{29}{3N}\Big)S_3(N) + \Big(\frac{19}{2} - 2(-1)^N\Big)S_4(N) + \big(-6+5(-1)^N\big)S_{-4}(N)$$
$$+ \Big(-\frac{2(-1)^N(9N+5)}{N(N+1)} - \frac{2}{N}\Big)S_{2,1}(N) + \big(20+2(-1)^N\big)S_{2,-2}(N) + \big(-17+13(-1)^N\big)S_{3,}$$
$$- \frac{8(-1)^N(2N+1)+4(9N+1)}{N(N+1)}S_{-2,1}(N) - \big(24+4(-1)^N\big)S_{-3,1}(N) + \big(3-5(-1)^N\big)S_{2,1,}$$
$$+ 32S_{-2,1,1}(N) + \left(\frac{3}{2}S_1(N)^2 - \frac{3S_1(N)}{N} + \frac{3}{2}(-1)^N S_{-2}(N)\right)\zeta(2)$$

$\boxed{F_0(N)} =$ (using `Sigma.m`, `EvaluateMultiSums.m` and J. Ablinger's `HarmonicSums.m` package)

$$\frac{7}{12}S_1(N)^4 + \frac{(17N+5)S_1(N)^3}{3N(N+1)} + \Big(\frac{35N^2-2N-5}{2N^2(N+1)^2} + \frac{13S_2(N)}{2} + \frac{5(-1)^N}{2N^2}\Big)S_1(N)^2$$

$$+\Big(-\frac{4(13N+5)}{N^2(N+1)^2} + \Big(\frac{4(-1)^N(2N+1)}{N(N+1)} - \frac{13}{N}\Big)S_2(N) + \Big(\frac{29}{3} - (-1)^N\Big)S_3(N)$$

$$+\big(2+2(-1)^N\big)S_{2,1}(N) - 28S_{-2,1}(N) + \frac{20(-1)^N}{N^2(N+1)}\Big)S_1(N) + \Big(\frac{3}{4} + (-1)^N\Big)S_2(N)^2$$

$$-2(-1 \ldots -1)^N\big)S_1(N) + \frac{4(-1)^N}{N \ldots}\Big)$$

$$+\Big(\frac{(-1 \ldots}{2N \ldots} \ldots N)^2 + \Big(\frac{8(-1)^N}{N(\ldots}$$

$$+\frac{4(3N \ldots}{N(N \ldots} \ldots 6(-1)^N\big)S_2(N) - \frac{\ldots}{N(N+1)}\Big)$$

$$+\Big(\frac{(-1)^N \ldots}{N(N+1)} - \frac{\ldots}{3N}\Big)S_3(N) + \Big(\frac{\ldots}{2} - 2(-1)^N\Big)S_4(N) + \big(-6+5(-1)^N\big)S_{-4}(N)$$

$$+\Big(-\frac{2(-1)^N(9N+5)}{N(N+1)} - \frac{2}{N}\Big)S_{2,1}(N) + \big(20+2(-1)^N\big)S_{2,-2}(N) + \big(-17+13(-1)^N\big)S_{3,\ldots}$$

$$-\frac{8(-1)^N(2N+1)+4(9N+1)}{N(N+1)}S_{-2,1}(N) - \big(24+4(-1)^N\big)S_{-3,1}(N) + \big(3-5(-1)^N\big)S_{2,1,\ldots}$$

$$+32S_{-2,1,1}(N) + \left(\frac{3}{2}S_1(N)^2 - \frac{3S_1(N)}{N} + \frac{3}{2}(-1)^N S_{-2}(N)\right)\zeta(2)$$

$$S_{-2,1,1}(N) = \sum_{i=1}^{N}\frac{(-1)^i\sum_{j=1}^{i}\frac{\sum_{k=1}^{j}\frac{1}{k}}{j}}{i^2}$$

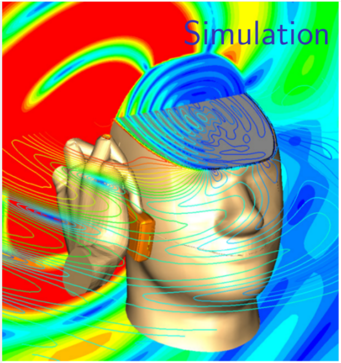
$$\zeta(2) = \sum_{k=1}^{\infty}\frac{1}{k^2}$$

Symbolic Special Functions Manipulation: Application in Electromagnetic Wave Simulation

Collaboration with company CST (Computer Simulation Technology)

Partner:	Joachim Schöberl (TU Vienna)
RISC:	Christoph Koutschan (now RICAM)
	Peter Paule

Simulation of electromagnetic waves



www.cst.com

- joint work by Joachim Schöberl (RWTH Aachen), Peter Paule and Christoph Koutschan (RISC)
- wide range of applications in constructing antennas, mobile phones, etc.
- merchandised by the company CST (Computer Simulation Technology)
- simulation with finite element methods
- significant contributions from Symbolic Computation using CK's package `HolonomicFunctions`
- symbolically derived formulae allow a considerable speed-up
- method is planned to be registered as a patent

Mathematical and physical background

Simulate the propagation of electromagnetic waves using the Maxwell equations

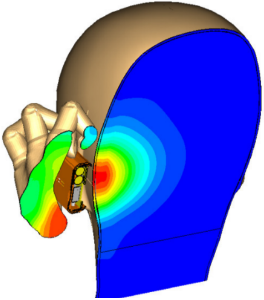
$$\frac{\mathrm{d}H}{\mathrm{d}t} = \operatorname{curl} E, \quad \frac{\mathrm{d}E}{\mathrm{d}t} = -\operatorname{curl} H$$

where H and E are the magnetic and the electric field respectively.

Define basis functions (in 2D) in order to approximate the solution:

$$\varphi_{i,j}(x,y) := (1-x)^i P_j^{(2i+1,0)}(2x-1) P_i\left(\frac{2y}{1-x} - 1\right)$$

Basis functions in 3D are more involved.



Results

In order to speed up the numerical computations, certain relations for the basis functions $\varphi_{i,j}(x,y)$ are needed.

Using HolonomicFunctions relations like the following can easily be derived:

$$\begin{aligned}
&2(i+2j+5)(2i+2j+7)\tfrac{\mathrm{d}}{\mathrm{d}x}\varphi_{i,j+1}(x,y)\\
&+(2i+1)(i+2j+1)\tfrac{\mathrm{d}}{\mathrm{d}x}\varphi_{i,j+2}(x,y)\\
&-(j+3)(i+2j+5)\tfrac{\mathrm{d}}{\mathrm{d}x}\varphi_{i,j+3}(x,y)\\
&+(j+1)(2i+2j+7)\tfrac{\mathrm{d}}{\mathrm{d}x}\varphi_{i+1,j}(x,y)\\
&-2(2i+3)(i+j+3)\tfrac{\mathrm{d}}{\mathrm{d}x}\varphi_{i+1,j+1}(x,y)\\
&+(i+2j+5)(2i+2j+5)\tfrac{\mathrm{d}}{\mathrm{d}x}\varphi_{i+1,j+2}(x,y)=\\
&2(i+j+4)(2i+2j+5)(2i+2j+7)\varphi_{i,j+2}(x,y)\\
&+2(i+j+2)(i+2j+5)(2i+2j+7)\varphi_{i+1,j+1}(x,y)
\end{aligned}$$

Much bigger formulae in the 3D case! Some efforts were needed to compute them.

Method

- basis functions $\varphi_{i,j}(x, y)$ are composed of functions that are holonomic and ∂-finite, i.e., hypergeometric expressions and orthogonal polynomials (Legendre and Jacobi)
- differential equations and recurrence relations for these objects are known
- symbolic algorithms deliver relations for $\varphi_{i,j}(x, y)$
- compute a Gröbner basis for the ideal of such relations
- search in the ideal for relations of the desired form
- all the above steps can be performed automatically by `HolonomicFunctions`

The RISC Software Company: the applied branch of RISC

Industrial Applications

CEO: Wolfgang Freiseisen
50 employees

About RISC Software GmbH

RISC Software GmbH

RISC Institute

- basic research in **Symbolic Computation**
- chair: Peter Paule
- founder (1987): Bruno Buchberger
- 50 employees (incl. PhD students)

RISC Company

- **Software Development**
- applied research in *Algorithmic Mathematics*
- technology transfer
- 50 employees

Ownership Structure

- 80% Johannes Kepler University
- 20% State Upper Austria (UAR GmbH)

Business Units

- ISA
- Logistics Inf.
- Medical Inf.
- Advanced Computing Technologies

JOHANNES KEPLER UNIVERSITY LINZ | JKU

1989 Working group for industrial applications at RISC Institute

1990 Foundation of Softwarepark Hagenberg under the management of RISC

1992 Foundation RISC Software GmbH (Prof. Bruno Buchberger)

1995 RISCSW specializes in software for logistics and production planning

2004 RISC Software GmbH becomes a 100%-subsidiary of JKU

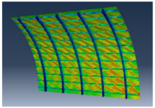
2008 Incorporation of department Medical Informatics (State Upper Austria 20%)

2013 Headcount 52; annual turnover ca MEUR 4,5

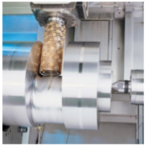
Industrial Software Applications

RISC Software GmbH

- Simulation and Design
  - virtual product development
  - engineering workflow management
    Example: TRUMPF bending machines



- Manufacturing Processes and Control Systems
  - simulation of machining processes (e.g. NC programs)
  - geometric modeling and visualization
    Example: CrashGuard for WFL MillTurn

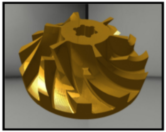


Software Development

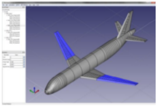
RISC Software GmbH

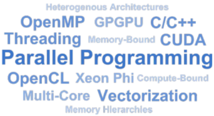
HANNES KEPLER IVERSITY LINZ | JKU

- Computational Mathematics
  - Numerical and symbolic computation
  - computational geometry and visualization



- Software Engineering
  - algorithm engineering
  - parallel computing





LAGRANGE – Multidisciplinary Structural Optimization Tool

- Software system
  1984 (Airbus/EADS)
  since 2003: RISCSW coop.
  since 2009: main development partner
  of Airbus D&S for LAGRANGE

Lagrange
Multidisciplinary Structural Optimisation System

- RISCSW contributions
  - *efficient, parallel algorithms*
    (e.g. solution of linear systems with very special structures)
  - *system architecture and development process*
    (test management, etc.)

RISC Software GmbH

HANNES KEPLER
NIVERSITY LINZ | JKU

Appendix: Subalgebra Membership Algorithm

Application: $\mathbb{C}$-subalgebras of $\mathbb{C}[z]$

GIVEN: $f_1, f_2, f_3 \in \mathbb{C}[z]$ with

$$\begin{aligned} t := f_1 &= z^6 + a\, z^5 + \dots, \\ f_2 &= z^9 + b\, z^8 + \dots, \\ f_3 &= z^{20} + c\, z^{19} + \dots; \end{aligned}$$

FIND: $g_1, \dots, g_k \in \mathbb{C}[z]$ such that

$$\begin{aligned} \mathbb{C}[f_1, f_2, f_3] &= \langle g_1, \dots, g_k \rangle_{\mathbb{C}[t]} \\ &= \mathbb{C}[t] + \mathbb{C}[t]\, g_1 + \dots + \mathbb{C}[t]\, g_k. \end{aligned}$$

Application: $\mathbb{C}$-subalgebras of $\mathbb{C}[z]$

GIVEN: $f_1, f_2, f_3 \in \mathbb{C}[z]$ with

$$\begin{aligned} t := f_1 &= z^6 + a\,z^5 + \dots, \\ f_2 &= z^9 + b\,z^8 + \dots, \\ f_3 &= z^{20} + c\,z^{19} + \dots; \end{aligned}$$

FIND: $g_1, \dots, g_k \in \mathbb{C}[z]$ such that

$$\begin{aligned} \mathbb{C}[f_1, f_2, f_3] &= \langle g_1, \dots, g_k \rangle_{\mathbb{C}[t]} \\ &= \mathbb{C}[t] + \mathbb{C}[t]\,g_1 + \dots + \mathbb{C}[t]\,g_k. \end{aligned}$$

KEY IDEA:

$$z^n + \dots = z^{u_i + \ell \cdot 6} + \dots = (z^{a_i \cdot 9 + b_i \cdot 20} + \dots)\,t^\ell + \dots.$$

KEY PROCEDURE

GIVEN: $f_1, f_2, f_3 \in \mathbb{C}[z]$ with

$$M = \langle \deg(f_1), \deg(f_2), \deg(f_3) \rangle = \langle 6, 9, 20 \rangle;$$

select $t := f_1$ and define $m := \deg(t) = 6$;

KEY PROCEDURE

GIVEN: $f_1, f_2, f_3 \in \mathbb{C}[z]$ with

$$M = \langle \deg(f_1), \deg(f_2), \deg(f_3) \rangle = \langle 6, 9, 20 \rangle;$$

select $t := f_1$ and define $m := \deg(t) = 6$;

FIND: $g_1, \ldots, g_5 \in \mathbb{C}[f_1, f_2, f_3]$ such that

$$\mathbb{C}[f_1, f_2, f_3] = \mathbb{C}[t, f_2, f_3] = \mathbb{C}[t, g_1, \ldots, g_5]$$

and

$$\{\deg(g_1), \ldots, \deg(g_5)\} = \{u_1, \ldots, u_5\} \equiv \{1, \ldots, 5\} \pmod{6}.$$

How to determine the g_i?

How to determine the g_i?

$[M]_0 = \{0, 6, 12, \dots\}$, and $t := f_1 = z^6 + a\,z^5 + \cdots$.

$[M]_1 = \{\not{1}, \not{7}, \not{13}, \not{19}, \not{25}, \not{31}, \not{37}, \not{43}, 49, 54, \dots\}$, define in view of $u_1 = 49 = \textcircled{9} + 2\,\textcircled{20}$,

$$g_1 := f_2 f_3^2 = z^{u_1} + \cdots .$$

$[M]_2 = \{\not{2}, \not{8}, \not{14}, 20, 26, \dots\}$, define in view of $u_2 = \textcircled{20}$,

$$g_2 := f_3 = z^{u_2} + \cdots .$$

$[M]_3 = \{\not{3}, 9, 15, \dots\}$, define in view of $u_3 = \textcircled{9}$,

$$g_3 := f_2 = z^{u_3} + \cdots .$$

$[M]_4 = \{\not{4}, \not{10}, \not{16}, \not{22}, \not{28}, \not{34}, 40, 46, \ldots\}$, define in view of

$u_4 = 40 = 2 \cdot \textcircled{20}$,

$$g_4 := f_3^2 = z^{u_4} + \cdots .$$

$[M]_5 = \{\not{5}, \not{11}, \not{17}, \not{23}, 29, 35, \ldots\}$, define in view of

$u_5 = 29 = \textcircled{9} + \textcircled{20}$,

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SUMMARY. We determined the $g_i \in \mathbb{C}[f_1, f_2, f_3]$ as

$$\begin{aligned} g_1 &:= f_2 f_3^2 = z^{u_1} + \cdots , \\ g_2 &:= f_3 = z^{u_2} + \cdots , \\ g_3 &:= f_2 = z^{u_3} + \cdots , \\ g_4 &:= f_3^2 = z^{u_4} + \cdots , \\ g_5 &:= f_2 f_3 = z^{u_5} + \cdots , \end{aligned}$$

such that

$$\mathbb{C}[f_1, f_2, f_3] = \mathbb{C}[t, g_1, \dots g_5] \text{ and } \deg(g_i) \equiv i \pmod 6.$$

Example. $f = z^{62} + \cdots \in \mathbb{C}[f_1, f_2, f_3]$:

$$62 = \textcircled{20} + 7\,\textcircled{6} = u_2 + 7\,\textcircled{6};$$

recalling $g_2 := f_3 = z^{u_2} + \cdots$ one has,

$$f - g_2\, t^7 = c_1\, z^{61} + \cdots .$$

$$61 = \textcircled{49} + 2\,\textcircled{6} = u_1 + 2\,\textcircled{6};$$

recalling $g_1 := f_2 f_3^2 = z^{u_1} + \cdots$ one has,

$$f - g_2\, t^7 - c_1\, g_1\, t^2 = c_2\, z^{60} + \cdots .$$

BUT:

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BUT: For instance, it could be that

$$f - g_2\, t^7 - c_1\, g_1\, t^2 = z^4 + \cdots \in \mathbb{C}[f_1, f_2, f_3].$$

Example (contd). $f = z^{62} + \cdots \in \mathbb{C}[f_1, f_2, f_3]$:

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NOTE.

$$4 \notin \langle 6, 9, 20 \rangle = \langle \deg(f_1), \deg(f_2), \deg(f_3) \rangle$$

BUT

$$4 \in \langle \deg(g) : g \in \mathbb{C}[f_1, f_2, f_3] \rangle.$$

SOLUTION.

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$$4 \in \langle \deg(g) : g \in \mathbb{C}[f_1, f_2, f_3] \rangle.$$

SOLUTION. We repeat the KEY PROCEDURE

$$\{f_1, f_2, f_3\} \rightsquigarrow \{g_1, g_2, g_3, g_4, g_5\}$$

by iteratively joining products:

$$\{g_1, g_2, g_3, g_4, g_5\} \cup \{g_i g_j\} \rightsquigarrow \{h_1, h_2, h_3, h_4, h_5\},$$

$$\{h_1, h_2, h_3, h_4, h_5\} \cup \{h_i h_j\} \rightsquigarrow \{\ldots\}, \text{ a.s.o.}$$

DECISION PROCEDURE: $\mathbb{C}$-subalgebra membership

GIVEN: f and $f_1, \ldots, f_n$ from $\mathbb{C}[z]$;

FIND: polynomials $p_0, p_1, \ldots, p_k \in \mathbb{C}[z]$ such that

$$\begin{aligned} f &= p_0(t) + p_1(t)\, g_1 + \cdots + p_k(t)\, g_k \\ &\in \langle 1, g_1, \ldots, g_k \rangle_{\mathbb{C}[t]} = \mathbb{C}[f_1, \ldots, f_n] \end{aligned}$$

APPLICATION.

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APPLICATION. With this procedure the right hand sides of Ramanujan identities can be found automatically: just apply it to given $f \in M^{\infty}(N)$ and $f_1, \ldots, f_n \in E^{\infty}(N)$. In this context, $z = \frac{1}{q}$ because of possible poles sitting at $[\infty]$; the f_j can be treated as formal Laurent series like

$$\begin{aligned} f_1 &= \frac{a_6}{q^6} + \frac{a_5}{q^5} + \cdots, \\ f_2 &= \frac{b_9}{q^9} + \frac{b_8}{q^8} + \cdots, \\ f_3 &= \frac{c_{20}}{q^{20}} + \frac{c_{19}}{q^{19}} + \cdots, \text{ etc.} \end{aligned}$$

Example. Radu's "Ramanujan-Kolberg" package in STEP 1 computes that:

$$q^{-1}\frac{E(q)^8}{E(q^7)^7}\sum_{n=0}^{\infty}p(7n+5)q^n \in M^{\infty}(7).$$

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In STEP 4 the package finds Ramanujan's identity:

$$q^{-1}\frac{E(q)^8}{E(q^7)^7}\sum_{n=0}^{\infty}p(7n+5)q^n = 49\cdot 1 + 7\cdot f_1.$$