On non-bipartite distance-regular graphs with valency k and smallest eigenvalue not larger than $-k/2$

J. Koolen*

*School of Mathematical Sciences
USTC
(Based on joint work with Zhi Qiao)

CoCoA, July, 2015

Outline

Outline

Defintion

Graph: $\Gamma = (V, E)$ where V vertex set, $E \subseteq \binom{V}{2}$ edge set.

- All graphs in this talk are simple.
- $x \sim y$ if $xy \in E$.
- $x \not\sim y$ if $xy \notin E$.
- $d(x, y)$: length of a shortest path connecting x and y.
- $D(\Gamma)$ diameter (max distance in Γ)

Distance-regular graphs

Definition

- $\Gamma_i(x) := \{y \mid d(x,y) = i\}$

Distance-regular graphs

Definition

- $\Gamma_i(x) := \{y \mid d(x,y) = i\}$

Definition

- A connected graph Γ is called **distance-regular** (DRG) if there are numbers a_i, b_i, c_i $(0 \leq i \leq D = D(\Gamma))$ s.t. if $d(x,y) = j$ then
  - $\#\Gamma_1(y) \cap \Gamma_{j-1}(x) = c_j$
  - $\#\Gamma_1(y) \cap \Gamma_j(x) = a_j$
  - $\#\Gamma_1(y) \cap \Gamma_{j+1}(x) = b_j$

Properties

Γ: a distance-regular graph with diameter D.

- Γ is b_0-regular. ($k := b_0$ is called its valency).

Properties

Γ: a distance-regular graph with diameter D.

- Γ is b_0-regular. ($k := b_0$ is called its valency).
- $1 = c_1 \leq c_2 \leq \ldots \leq c_D$.
- $b_0 \geq b_1 \geq \ldots \geq b_{D-1}$.

Outline

Hamming graphs

Definition

- $q \geq 2$, $n \geq 1$ integers.
- $Q = \{1, \ldots, q\}$
- Hamming graph $H(n, q)$ has vertex set Q^n
- $\mathbf{x} \sim \mathbf{y}$ if they differ in exactly one position.
- Diameter equals n.

Hamming graphs

Definition

- $q \geq 2$, $n \geq 1$ integers.
- $Q = \{1, \ldots, q\}$
- Hamming graph $H(n, q)$ has vertex set Q^n
- $\mathbf{x} \sim \mathbf{y}$ if they differ in exactly one position.
- Diameter equals n.
- $H(n, 2) = n$-cube.
- DRG with $c_i = i$.

Hamming graphs

Definition

- $q \geq 2$, $n \geq 1$ integers.
- $Q = \{1, \dots, q\}$
- Hamming graph $H(n, q)$ has vertex set Q^n
- $\mathbf{x} \sim \mathbf{y}$ if they differ in exactly one position.
- Diameter equals n.
- $H(n, 2) = n$-cube.
- DRG with $c_i = i$.
- Gives an algebraic frame work to study codes, especially bounds on codes.
- For example the Delsarte linear programming bound and more recently the Schrijver bound.

Johnson graphs

Definition

- $1 \leq t \leq n$ integers.
- $N = \{1, \ldots, n\}$
- Johnson graph $J(n, t)$ has vertex set $\binom{N}{t}$
- $A \sim B$ if $\#A \cap B = t - 1$.
- $J(n, t) \approx J(n, n - t)$, diameter $\min(t, n - t)$.
- DRG with $c_i = i^2$.
- Gives an algebraic frame work to study designs.

More examples

- Most of the known families of distance-regular graphs come from classical objects, for example:

More examples

- Most of the known families of distance-regular graphs come from classical objects, for example:
  - Hamming graphs,
  - Johnson graphs,
  - Grassmann graphs,
  - bilinear forms graphs,
  - sesquilinear forms graphs,
  - quadratic forms graphs,
  - dual polar graphs (The vertices are the maximal totally isotropic subspaces on a vectorspace over a finite field with a fixed (non-degenerate) bilinear form)

More examples

- Most of the known families of distance-regular graphs come from classical objects, for example:
  - Hamming graphs,
  - Johnson graphs,
  - Grassmann graphs,
  - bilinear forms graphs,
  - sesquilinear forms graphs,
  - quadratic forms graphs,
  - dual polar graphs (The vertices are the maximal totally isotropic subspaces on a vectorspace over a finite field with a fixed (non-degenerate) bilinear form)
- Distance-regular graphs gives a way to study these classical objects from a combinatorial view point.

Eigenvalues of graphs

- Let Γ be a graph.
- The **adjacency matrix** for Γ is the symmetric matrix A indexed by the vertices st. $A_{xy} = 1$ if $x \sim y$, and 0 otherwise.
- The eigenvalues of A are called the eigenvalues of Γ.

Eigenvalues of graphs

- Let Γ be a graph.
- The **adjacency matrix** for Γ is the symmetric matrix A indexed by the vertices st. $A_{xy} = 1$ if $x \sim y$, and 0 otherwise.
- The eigenvalues of A are called the eigenvalues of Γ.
- As A is a real symmetric matrix all its eigenvalues are real. We mainly will look at the smallest eigenvalue.

Outline

Examples

In this section, we study the non-bipartite distance-regular graphs with valency k and having a smallest eigenvalue not larger than $-k/2$.

Examples

In this section, we study the non-bipartite distance-regular graphs with valency k and having a smallest eigenvalue not larger than $-k/2$.

Examples

1. The odd polygons with valency 2;
2. The complete tripartite graphs $K_{t,t,t}$ with valency $2t$ at least 2;
3. The folded $(2D+1)$-cubes with valency $2D+1$ and diameter $D \geq 2$;
4. The Odd graphs with valency k at least 3;
5. The Hamming graphs $H(D,3)$ with valency $2D$ where $D \geq 2$;
6. The dual polar graphs of type $B_D(2)$ with $D \geq 2$;
7. The dual polar graphs of type ${}^2A_{2D-1}(2)$ with $D \geq 2$.

Conjecture

Conjecture

If $D > 0$ is large enough, and the smallest eigenvalue is not larger than $-k/2$, then Γ is a member of one of the seven families.

Outline

Valency Bound

Theorem

For any real number $1 > \alpha > 0$ and any integer $D \geq 2$, the number of coconnected (i.e. the complement is connected) non-bipartite distance-regular graphs with valency k at least two and diameter D, having smallest eigenvalue $\theta_{\min}$ not larger than $-\alpha k$, is finite.

Remarks

- Note that the regular complete t-partite graphs $K_{t \times s}$ (s, t positive integers at least 2) with valency $k = (t-1)s$ have smallest eigenvalue $-s = -k/(t-1)$.

Remarks

- Note that the regular complete t-partite graphs $K_{t\times s}$ (s, t positive integers at least 2) with valency $k = (t-1)s$ have smallest eigenvalue $-s = -k/(t-1)$.
- Note that there are infinitely many bipartite distance-regular graphs with diameter 3, for example the point-block incidence graphs of a projective plane of order q, where q is a prime power.

Remarks

- Note that the regular complete t-partite graphs $K_{t \times s}$ (s, t positive integers at least 2) with valency $k = (t-1)s$ have smallest eigenvalue $-s = -k/(t-1)$.
- Note that there are infinitely many bipartite distance-regular graphs with diameter 3, for example the point-block incidence graphs of a projective plane of order q, where q is a prime power.
- The second largest eigenvalue for a distance-regular graphs behaves quite differently from its smallest eigenvalue. For example $J(n, D)$ $n \geq 2D \geq 4$, has valency $D(n-D)$, and second largest eigenvalue $(n-D-1)(D-1)-1$. So for fixed diameter D, there are infinitely many Johnson graphs $J(n, D)$ with second largest eigenvalue larger then $k/2$.

Ingredients for the proof

Biggs' formula

- Let θ be an eigenvalue of Γ.
- Let $\mathbf{u}(\theta) = (u_0 = 1, u_1, \ldots, u_D)^T$ be the standard vector for θ.
- The u_i's satisfy: $c_i u_{i-1} + a_i u_i + b_i u_{i+1} = \theta u_i \quad (0 \leq i \leq D)$ where $u_{-1} = u_{D+1} = 0$.

Ingredients for the proof

Biggs' formula

- Let θ be an eigenvalue of Γ.
- Let $\mathbf{u}(\theta) = (u_0 = 1, u_1, \ldots, u_D)^T$ be the standard vector for θ.
- The u_i's satisfy: $c_i u_{i-1} + a_i u_i + b_i u_{i+1} = \theta u_i \quad (0 \leq i \leq D)$ where $u_{-1} = u_{D+1} = 0$.
- (Biggs' formula) $\text{mult}(\theta) = \frac{v}{\sum_{i=0}^{D} k_i u_i^2} \quad (v = \#V(\Gamma),$ $k_i = \#\Gamma_i(x))$

Ingedients for the proof, II

The next observation gives a bound on the valency k, given the multiplicity of an eigenvalue different from $\pm k$.

Ingedients for the proof, II

The next observation gives a bound on the valency k, given the multiplicity of an eigenvalue different from $\pm k$.

Godsil's observation

Let Γ be a coconnected distance-regular graph with valency k. Let m be the multiplicity of an eigenvalue of Γ distinct from $\pm k$. Then $k \leq (m-1)(m+2)/2$.

Idea for the proof

- Let θ be the smallest eigenvalue of Γ.
- Let $(u_0, u_1, \ldots, u_D)$ be the standard vector for θ. Then $(-1)^i u_i > 0$.

Idea for the proof

- Let θ be the smallest eigenvalue of Γ.
- Let $(u_0, u_1, \ldots, u_D)$ be the standard vector for θ. Then $(-1)^i u_i > 0$.
- Now for all i either $|u_i|$ is large or c_i is large.

Idea for the proof

- Let θ be the smallest eigenvalue of Γ.
- Let $(u_0, u_1, \ldots, u_D)$ be the standard vector for θ. Then $(-1)^i u_i > 0$.
- Now for all i either $|u_i|$ is large or c_i is large.
- Both will help to get a good bound for the multiplicity using Biggs' formula. The second one will give an upper bound for the vertices.

Idea for the proof

- Let θ be the smallest eigenvalue of Γ.
- Let $(u_0, u_1, \ldots, u_D)$ be the standard vector for θ. Then $(-1)^i u_i > 0$.
- Now for all i either $|u_i|$ is large or c_i is large.
- Both will help to get a good bound for the multiplicity using Biggs' formula. The second one will give an upper bound for the vertices.
- Then we use Godsil's observation to bound k.

Outline

Coconnected

Let Γ be a distance-regular graph with valency $k \geq 2$ and smallest eigenvalue $\lambda_{\min} \leq -k/2$. It is easy to see that if the graph is coconnected then $a_1 \leq 1$.

Now we give the classification for diameter 2.

Now we give the classification for diameter 2.

Diameter 2

1. The pentagon with intersection array $\{2, 1; 1, 1\}$;
2. The Petersen graph with intersection array $\{3, 2; 1, 1\}$;
3. The folded 5-cube with intersection array $\{5, 4; 1, 2\}$;
4. The 3×3-grid with intersection array $\{4, 2; 1, 2\}$;
5. The generalized quadrangle $GQ(2, 2)$ with intersection array $\{6, 4; 1, 3\}$;
6. The generalized quadrangle $GQ(2, 4)$ with intersection array $\{10, 8; 1, 5\}$;
7. A complete tripartite graph $K_{t,t,t}$ with $t \geq 2$, with intersection array $\{2t, t-1; 1, 2t\}$;

Now we give the classification for diameter 2.

Diameter 2

1. The pentagon with intersection array $\{2, 1; 1, 1\}$;
2. The Petersen graph with intersection array $\{3, 2; 1, 1\}$;
3. The folded 5-cube with intersection array $\{5, 4; 1, 2\}$;
4. The 3×3-grid with intersection array $\{4, 2; 1, 2\}$;
5. The generalized quadrangle $GQ(2, 2)$ with intersection array $\{6, 4; 1, 3\}$;
6. The generalized quadrangle $GQ(2, 4)$ with intersection array $\{10, 8; 1, 5\}$;
7. A complete tripartite graph $K_{t,t,t}$ with $t \geq 2$, with intersection array $\{2t, t-1; 1, 2t\}$;

No suprises.

Outline

Bound on c_2

- Although we found a valency bound in general for distance-regular graphs with fixed diameter D and smallest eigenvalue not larger than $-k/2$, this bound is not good enough to classify those of diameter 3.

Bound on c_2

- Although we found a valency bound in general for distance-regular graphs with fixed diameter D and smallest eigenvalue not larger than $-k/2$, this bound is not good enough to classify those of diameter 3.
- Using the representation theory of a distance-regular with respect to its smallest eigenvalue, we obtained a bound on c_2.

Bound on c_2

- Although we found a valency bound in general for distance-regular graphs with fixed diameter D and smallest eigenvalue not larger than $-k/2$, this bound is not good enough to classify those of diameter 3.
- Using the representation theory of a distance-regular with respect to its smallest eigenvalue, we obtained a bound on c_2.

Lemma

Let Γ be a non-bipartite distance-regular graph with diameter $D \geq 3$ and valency $k \geq 2$. If the smallest eigenvalue of Γ, $\theta_{\min}$, is at most $-k/2$, then $a_1 \leq 1$ and $c_2 \leq 5 + a_1$.

Diameter 3 and triangle-free

In the following we give the classification of distance-regular graphs with diameter 3 valency $k \geq 2$ with smallest eigenvalue not larger than $-k/2$.

Diameter 3 and triangle-free

In the following we give the classification of distance-regular graphs with diameter 3 valency $k \geq 2$ with smallest eigenvalue not larger than $-k/2$. Because of the above lemma we obtained that the multiplicity of the smallest eigenvalue is at most 64 and hence the valency is at most 64 if $a_1 = 0$.

Diameter 3 and triangle-free

In the following we give the classification of distance-regular graphs with diameter 3 valency $k \geq 2$ with smallest eigenvalue not larger than $-k/2$. Because of the above lemma we obtained that the multiplicity of the smallest eigenvalue is at most 64 and hence the valency is at most 64 if $a_1 = 0$.
Our result:

Diameter 3

1. The 7-gon, with intersection array $\{2, 1, 1; 1, 1, 1\}$;
2. The Odd graph with valency 4, O_4, with intersection array $\{4, 3, 3; 1, 1, 2\}$;
3. The Sylvester graph with intersection array $\{5, 4, 2; 1, 1, 4\}$;
4. The second subconstituent of the Hoffman-Singleton graph with intersection array $\{6, 5, 1; 1, 1, 6\}$;
5. The Perkel graph with intersection array $\{6, 5, 2; 1, 1, 3\}$;

Diameter 3 and triangle-free, II

Theorem continued

1. The folded 7-cube with intersection array $\{7, 6, 5; 1, 2, 3\}$;
2. A possible distance-regular graph with intersection array $\{7, 6, 6; 1, 1, 2\}$;
3. A possible distance-regular graph with intersection array $\{8, 7, 5; 1, 1, 4\}$;
4. The truncated Witt graph associated with M_{23} with intersection array $\{15, 14, 12; 1, 1, 9\}$;
5. The coset graph of the truncated binary Golay code with intersection array $\{21, 20, 16; 1, 2, 12\}$;

Diameter 3 and triangle-free, II

Theorem continued

1. The folded 7-cube with intersection array $\{7, 6, 5; 1, 2, 3\}$;
2. A possible distance-regular graph with intersection array $\{7, 6, 6; 1, 1, 2\}$;
3. A possible distance-regular graph with intersection array $\{8, 7, 5; 1, 1, 4\}$;
4. The truncated Witt graph associated with M_{23} with intersection array $\{15, 14, 12; 1, 1, 9\}$;
5. The coset graph of the truncated binary Golay code with intersection array $\{21, 20, 16; 1, 2, 12\}$;

So this means that for diameter 3 and triangle-free, we obtain quite a few more examples, then the members of the three families.

Diameter 3 and $a_1 \neq 0$

In this case, we obtain the following classification.

Diameter 3 and $a_1 \neq 0$

In this case, we obtain the following classification.

Theorem

1. The line graph of the Petersen graph with intersection array $\{4, 2, 1; 1, 1, 4\}$;
2. The generalized hexagon $GH(2, 1)$ with intersection array $\{4, 2, 2; 1, 1, 2\}$;
3. The Hamming graph $H(3, 3)$ with intersection array $\{6, 4, 2; 1, 2, 3\}$;
4. One of the two generalized hexagons $GH(2, 2)$ with intersection array $\{6, 4, 4; 1, 1, 3\}$;
5. One of the two distance-regular graphs with intersection array $\{8, 6, 1; 1, 3, 8\}$;
6. The regular near hexagon $B_3(2)$ with intersection array $\{14, 12, 8; 1, 3, 7\}$;

Theorem continued

1. The generalized hexagon $GH(2,8)$ with intersection array $\{18, 16, 16; 1, 1, 9\}$;
2. The regular near hexagon on 729 vertices related to the extended ternary Golay code with intersection array $\{24, 22, 20; 1, 2, 12\}$;
3. The Witt graph associated to M_{24} with intersection array $\{30, 28, 24; 1, 3, 15\}$;
4. The regular near hexagon ${}^2A_5(2)$ with intersection array $\{42, 40, 32; 1, 5, 21\}$.

Theorem continued

1. The generalized hexagon $GH(2,8)$ with intersection array $\{18, 16, 16; 1, 1, 9\}$;
2. The regular near hexagon on 729 vertices related to the extended ternary Golay code with intersection array $\{24, 22, 20; 1, 2, 12\}$;
3. The Witt graph associated to M_{24} with intersection array $\{30, 28, 24; 1, 3, 15\}$;
4. The regular near hexagon ${}^2A_5(2)$ with intersection array $\{42, 40, 32; 1, 5, 21\}$.

We also classified diameter 4 and $a_1 \neq 0$. The classification looks quite similar to the diameter 3 case.

Outline

Now we will use the classification of the diameter 3 case to determine the 3-chromatic distance-regular graphs.

Now we will use the classification of the diameter 3 case to determine the 3-chromatic distance-regular graphs.

- A proper coloring with t colors of a graph Γ is a map $c : v(\Gamma) \to \{1, 2, \ldots, t\}$ (t is a positive number) such that $c(x) \neq c(y)$ for any edge xy.
- The chromatic number of Γ denoted by $\chi(\Gamma)$ is the minimal t such that there exists a proper coloring of Γ with t colors. We also say that such a graph is $\chi(\Gamma)$-chromatic.

Now we will use the classification of the diameter 3 case to determine the 3-chromatic distance-regular graphs.

- A proper coloring with t colors of a graph Γ is a map $c : v(\Gamma) \to \{1, 2, \ldots, t\}$ (t is a positive number) such that $c(x) \neq c(y)$ for any edge xy.
- The chromatic number of Γ denoted by $\chi(\Gamma)$ is the minimal t such that there exists a proper coloring of Γ with t colors. We also say that such a graph is $\chi(\Gamma)$-chromatic.
- An independent set of Γ is a set S of vertices such that there are no edges between them.
- If Γ has chromatic number χ and n vertices then Γ must have an independent set with at least n/χ vertices.

An useful result is the Hoffman bound.

Hoffman bound

Let G be a k-regular graph with n vertices and with smallest eigenvalue $\theta_{\min}$. Let S be an independent set of Γ with s vertices. Then

$$s \leq \frac{n}{1 + \frac{k}{-\theta_{\min}}}.$$

- Let Γ be a 3-chromatic distance-regular graph with n vertices.
- Then Γ must have an independent set of size at least $n/3$ and by the Hoffman bound we find that the smallest eigenvalue of Γ is at most $-k/2$.

- Let Γ be a 3-chromatic distance-regular graph with n vertices.
- Then Γ must have an independent set of size at least $n/3$ and by the Hoffman bound we find that the smallest eigenvalue of Γ is at most $-k/2$.
- Blokhuis et al. determined the 3-chromatic distance-regular graphs among the known examples.

- Let Γ be a 3-chromatic distance-regular graph with n vertices.
- Then Γ must have an independent set of size at least $n/3$ and by the Hoffman bound we find that the smallest eigenvalue of Γ is at most $-k/2$.
- Blokhuis et al. determined the 3-chromatic distance-regular graphs among the known examples.
- Combining their result with the classification of the distance-regular graph with valency $k \geq 2$, diameter 3 and smallest eigenvalue not larger than $-k/2$ we obtain:

Theorem

Let Γ be a 3-chromatic distance-regular graph with diameter 3. Then Γ is one of the following:

1. The 7-gon, with intersection array $\{2, 1, 1; 1, 1, 1\}$;
2. The Odd graph with valency 4, O_4, with intersection array $\{4, 3, 3; 1, 1, 2\}$;
3. The Perkel graph with intersection array $\{6, 5, 2; 1, 1, 3\}$;
4. The generalized hexagon $GH(2, 1)$ with intersection array $\{4, 2, 2; 1, 1, 2\}$;
5. The Hamming graph $H(3, 3)$ with intersection array $\{6, 4, 2; 1, 2, 3\}$;
6. The regular near hexagon on 729 vertices related to the extended ternary Golay code with intersection array $\{24, 22, 20; 1, 2, 12\}$.

Hamming graph

- To show that the Hamming graph $H(D, q)$ is q-chromatic, represent the alphabet of size q by the integers mod q.
- Give a vertex color i ($i = 0, 1, \ldots, q-1$) if the sum of its entries equal i mod q.
- On the other hand $H(D, q)$ has a complete subgraph with q vertices.

Outline

Some open problems:

- Determine the 4-chromatic distance-regular graphs of diameter 3.

Some open problems:

- Determine the 4-chromatic distance-regular graphs of diameter 3.
- Determine the non-bipartite distance-regular graphs with diameter 3, valency k, such that its smallest eigenvalue is not larger than $-k/3$.

Some open problems:

- Determine the 4-chromatic distance-regular graphs of diameter 3.
- Determine the non-bipartite distance-regular graphs with diameter 3, valency k, such that its smallest eigenvalue is not larger than $-k/3$.
- Complete the classification of diameter 4 with smallest eigenvalue not larger than $-k/2$.

Some open problems:

- Determine the 4-chromatic distance-regular graphs of diameter 3.
- Determine the non-bipartite distance-regular graphs with diameter 3, valency k, such that its smallest eigenvalue is not larger than $-k/3$.
- Complete the classification of diameter 4 with smallest eigenvalue not larger than $-k/2$.
- Determine the distance-regular graphs with $a_1 = 1$ and smallest eigenvalue $-k/2$.

Thank you for your attention.