Self-complementary strongly regular graphs revisited

Ferenc Szöllősi
(joint work in part with Patric Östergård)
szoferi@gmail.com

Department of Communications and Networking, Aalto University

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What is this talk about?

This talk is about strongly regular graphs.

Definition:

A (v, k, λ, μ)-graph is strongly regular, if it is a k-regular graph of order v where two adjacent vertices have λ common neighbors, and two nonadjacent vertices have μ common neighbors.

The smallest open case with respect to v is the following.

Problem

Decide the existence of $\mathrm{SRG}(65, 32, 15, 16)$.

We outline two possible attacks against Problem.

Graphs with symmetries – tactical decomposition

We think of a graph as a well-chosen adjacency matrix A.

- Assume that A has an automorphism σ with m σ-orbits of size b_i;
- assume that A satisfies a quadratic equation $A^2 = \alpha I + \beta J + \gamma A$.

Then there is a reduced matrix R of size m for which

$$R^T \Delta R = \alpha\Delta + \beta r r^T + \gamma \Delta R$$

holds with $r = [b_i]^T$, and $\Delta = \mathrm{diag}(r)$.

$[R]_{i,j}$ is the row sum of the (i, j)-th block of A.

If σ is trivial, then $R = A$. If not, then R is a smaller matrix with entries $\{0, \ldots, \max b_i\}$.

R is also called orbit-matrix, or block-valency matrix.

Naive approach

- Classify the orbit matrices as a first step;
- Reconstruct the adjacency matrices as a second step.

This is a dead end.

Theorem[Behbahani, 2009]

Assume that a $\mathrm{SRG}(65, 32, 15, 16)$ has an automorphism of order q. Then the possible prime divisors of q are $2, 3$, or 5. Moreover, there exist exactly 36 orbit matrices corresponding to the automorphism 5 case.

Estimate (Behbahani): 10 million days (?!) to reconstruct the graphs from the orbit matrices.

Classifying the orbit matrices corresponding to the automorphism $2, 3$ cases is probably hopeless.

Attack1: Self-complementary strongly regular graphs

Definition

A graph is self-complementary, if it is isomorphic to its own complement.

Examples: Paley graphs.

Problem

Is there a self-complementary $\mathrm{SRG}(65, 32, 15, 16)$?

Answer: I still don't know, but at least this is something to aim for.

This condition gives rise to some highly structured orbit matrices.

Results on self-complementary graphs

Lemma

A self-complementary strongly regular graph is necessarily a conference graph.

Proof: A and $J - A - I$ must have the same parameters.

If A is self-complementary, then there is a complementing permutation (of the vertex set) σ mapping edges to non-edges.

Lemma

If σ is a complementing permutation, then it has at most one fixedpoint, and moreover, the length of every cycle is divisible by 4. Moreover, σ can be chosen in a way that all cycle lengths are 2-powers.

Remark: σ^2 is a non-trivial automorphism.

Example: Paley(13)

Self-complementarity leads to alternating circulant block structure;

Example: A self-complementary s.r.g of order 13: $A^2 = 3(J + I) - A$.

$$A = \left[\begin{array}{c|cccc|cccc|cccc}
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
\hline
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\
\hline
1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
\hline
1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0
\end{array}\right].$$

Example (ctd.)

Self-complementarity leads to ~~alternating~~ circulant block structure;

Example: A self-complementary s.r.g of order 13: $A^2 = 3(J + I) - A$.

$$A = \left[\begin{array}{c||cc|cc||cc|cc||cc|cc}
0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
\hline\hline
1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
\hline
0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
\hline\hline
1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
\hline\hline
1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
\hline
0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0
\end{array}\right].$$

Therefore tactical decomposition can be used!

Example (ctd. ctd.)

If A is the adjacency matrix of a self-complementary $\mathrm{SRG}(4t+1, 2t, t-1, t)$, then

$$A^2 = t(J+I) - A.$$

Assume that A has an $s \times s$ block partition for circulant blocks with orbit sizes $b_1, \dots, b_s$. Let $r := (b_1, b_2, \dots, b_s)^T$, and let $\Delta := \mathrm{diag}(r)$. Then

$$R^T \Delta R = t(\Delta + rr^T) - \Delta R.$$

Example (continued):

$$R = \left[\begin{array}{c|cc|cc|cc} 0 & 2 & 0 & 2 & 0 & 2 & 0 \\ \hline 1 & 0 & 1 & 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 2 & 1 & 0 & 1 \\ \hline 1 & 1 & 2 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 2 & 1 \\ \hline 1 & 1 & 0 & 1 & 2 & 0 & 1 \\ 0 & 2 & 1 & 0 & 1 & 1 & 1 \end{array}\right].$$

Every other column follows for free.

Mathon's results

Philosophy

- Self-complementarity lead to circulant structure;
- Block-valency matrices can be classified;
- Then the circulant matrices can be reconstructed.

Theorem

All self-complementary strongly regular graphs are known up to $n \leq 49$. (There are examples beyond the Paley-graphs.)

The main result of Mathon, however, is the discovery of *new* conference graphs of order 45 in this way, which have *higher* symmetries than what he found in 1978 by hand!

- "Computer trumps paper and pencil."

Remark: Mathon incorrectly claims certain results.

Some new results (Patent pending)

We have classified all orbit matrices related to self-complementary strongly regular graphs of order $n \leq 65$, except for two difficult cases.

- For $n = 53$ there are $3 + 9 + 13 + 103$ such matrices;
- ($n = 57$ is nonexistent)
- For $n = 61$ there are $9 + 1+???$ such matrices;
- For $n = 65$ there are $1 + 3+???$ such matrices;
- (It is possible to explore some cases of order $n = 85$.)

The cases depend on the length of the cycles of the complementing permutations.

Theorem

There does not exists self-complementary strongly regular graphs of order 65, except possible when the cycle length is $(1, 4, \ldots, 4)$.

Remark: There should be no problem reconstructing the graphs.

Attack2: Via conference matrices

Definition:

A conference matrix is an orthogonal $\{-1, 0, 1\}$-matrix with zero diagonal and ± 1 entries otherwise.

In particular, if C is a conference matrix of order n, then $CC^T = (n-1)I$.

Lemma:

If A is the adjacency matrix of a conference graph, B is the matrix A, bordered by a row and column of 1 (except for the diagonal entry), then $C := 2A - J + I$ is a conference matrix.

Example

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix} \rightsquigarrow B = \left[\begin{array}{c|ccccc} 0 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{array}\right].$$

$$C = \left[\begin{array}{c|ccccc} 0 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & 0 & 1 & -1 & -1 & 1 \\ 1 & 1 & 0 & 1 & -1 & -1 \\ 1 & -1 & 1 & 0 & 1 & -1 \\ 1 & -1 & -1 & 1 & 0 & 1 \\ 1 & 1 & -1 & -1 & 1 & 0 \end{array}\right].$$

Problem[Might be easy?]

What is the relation between the symmetries of A and C?

Results

Problem

Study the symmetries of conference matrices.

We considered conference matrices with fixedpoint-free automorphisms of order 3.

Theorem

- There exists conference matrices with the previous structure for $n \in \{6, 18, 30, 42\}$;
- Cases $n \in \{54, 66\}$ are open;
- Case $n = 78$ is impossible (as $n - 1$ is not the sum of two squares).

Remark: 54 is doable.
Remark: What about other automorphisms?

Summary: What happened during this talk?

- I recalled an existence-type problem on conference graphs.
- I recalled the tool of tactical decomposition.
- I combined it with the property of self-complementarity.
- I discovered some new exciting orbit matrices of order 65...,
- ...which all turned out to be a dead-end.
- I pointed out the related objects called conference matrices.
- I constructed some new conference matrices of low orders.
- More results to come.

Thank you...

...and see you at NORCOM2016 in Finland next year.