Rank 3 groups and partial linear spaces

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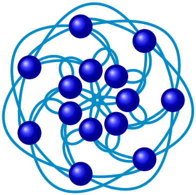
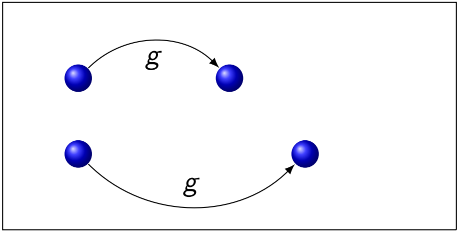
July 21, 2015



Permutation groups

- Over the years, many classes of permutation groups have been characterised and classified.
- A characterisation theorem describes what shape or structure the groups will have.
- A classification theorem lists the groups.
- I'm going to talk about some of these characterisation and classification theorems.

2-transitive groups



2-transitive groups

Theorem (Burnside, 1911)

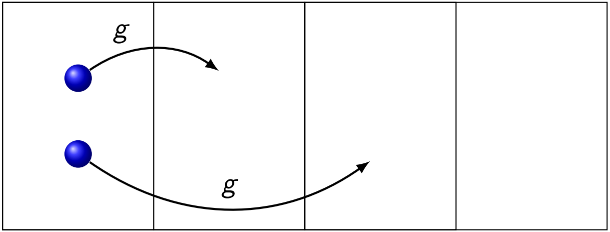
A finite 2-transitive permutation group is either affine or almost simple.

That is, if G is a finite 2-transitive group, then either:

- $G \leq \mathrm{AGL}(V)$ for some finite vector space V, or
- $S \trianglelefteq G \leq \mathrm{Aut}(S)$ for some finite simple group S.

This is a characterisation theorem for 2-transitive groups.

Primitive groups



Primitive groups

Theorem (The O'Nan-Scott Theorem, 1979)

A finite primitive group is one of the following types:

I. *Affine groups;*

II. *Almost simple groups;*

III. (a) *Simple diagonal action;*
 (b) *Product action;*
 (c) *Twisted wreath action.*

This is a characterisation theorem for primitive groups, with Burnside's Theorem occuring as a corollary.

Quasiprmitive groups

Definition (Quasiprimitive permutation group)

A permutation group is called *quasiprimitive* if all of its non-trivial normal subgroups are transitive.

Sorry, no neat picture this time.

Quasiprmitive groups

- All primitive groups are quasiprimitive.
- In the early days of group theory, quasiprimitive groups were sometimes called primitive groups.
    - A lot of early theorems for primitive groups are actually for quasiprimitive groups because of this.
- Quasiprimitive groups come up from time to time in Galois theory, and have found uses in graph theory.

Quasiprmitive groups

Theorem (Praeger, 1993)

A finite quasiprimitive group is one of the following types:

I. *Affine groups;*

II. *Almost simple groups;*

III. (a) *Simple diagonal action;*
   (b) *Product action;*
   (c) *Twisted wreath action.*

It looks identical to the O'Nan-Scott Theorem, because it pretty much is.

Innately transitive groups

Definition

A permutation group is *innately transitive* if it has a transitive minimal normal subgroup.

- All quasiprimitive groups are innately transitive.
- Like quasiprimitivity, innate transitivity doesn't have a neat picture.

Innately transitive permutation groups

- In his Ph.D., Bamberg developed the theory of innately transitive groups.
- A transitive minimal normal subgroup is called a *plinth*.
- Bamberg found that innately transitive groups can be characterised entirely by their plinth.

Innate triples

- Every innately transitive group induces a unique triple (K, φ, L), where:

    1. K is the plinth;
    2. φ is an epimorphism from a particular subgroup of K to the centraliser of K;
    3. L is the subgroup of $\mathrm{Aut}(K)$ induced by the conjugation action of the point-stabiliser in the innately transitive group.

- Further, K, φ and L are all that is needed to uniquely determine the original innately transitive group.

- These triples are called *innate triples*.

Innately transitive permutation groups

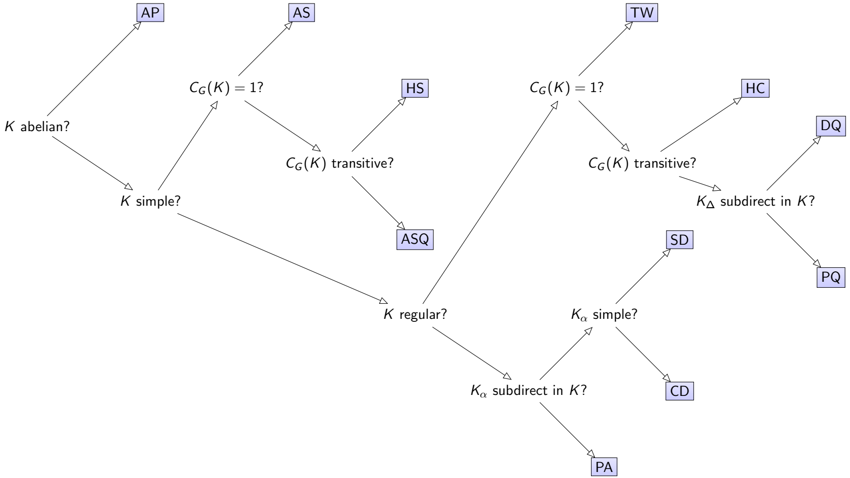
Theorem (Bamberg and Praeger, 2004)

A finite innately transitive group is one of the following types:

I. *Abelian plinth (AP);*
II. *Simple plinth (AS, HS, ASQ);*
III. *Regular plinth (TW, HC, DQ, PQ);*
IV. *Product action type (PA);*
V. *Diagonal type (SD, CD).*

A structure theorem for innately transitive groups, encompassing the O'Nan-Scott and Praeger theorems.

Subdivision of innately transitive groups by the plinth K



Rank has a million definitions, here's one

Definition

The *rank* of a transitive permutation group on a set Ω is equivalently:

1. the number of orbits the group has on $\Omega \times \Omega$;
2. the number of orbits a point-stabiliser has on Ω.

The first thing to note is that 2-transitive and rank 2 are the same thing.

Classifications

- Following from the Classification of Finite Simple groups, and some representation theory, all finite rank 2 groups have been classified.
    - Examples include $\mathrm{PSL}(n, q)$, $\mathrm{Sp}(2d, 2)$, $\mathrm{Sz}(q)$, $\mathrm{AGL}(n, q)$, $\mathrm{PSL}(2, 8)$...

- Work by Bannai (1972), Kantor and Liebler (1981-1982), and Liebeck and Saxl (1986-1987) produced a classification of finite primitive rank 3 groups. They fall into three families:
    1. Almost simple;
    2. Grid;
    3. Affine.

Classifications of imprimitive rank 3 groups

- The finite quasiprimitive rank 3 permutation groups were classified in 2011 (Devillers et al.).
- The classification was possible due to finite imprimitive quasiprimitive rank 3 groups being isomorphic to almost simple rank 2 groups.
  - Imprimitive quasiprimitive rank 3 groups are just rank 2 groups doing a different dance.
- All examples are in the PSL families, or M_{11}.

Classifications of imprimitive rank 3 groups

- I have been working with Bamberg, Devillers and Praeger on classifying the innately transitive rank 3 groups.
- Innately transitive rank 3 groups also come from almost simple rank 2 groups, but without the nice isomorphism.
- Thanks to Bamberg's framework for innately transitive groups, we have developed the machinery to classify innately transitive rank 3 groups.
- So far, the only examples have plinth PSL, A_5 or M_{11}.

A partial linear space is...

Definition

A partial linear space is a geometry of points and lines where no two points lie on more than one common line.

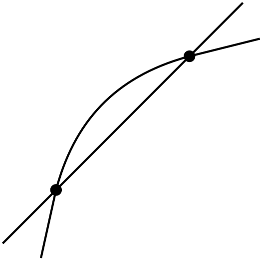
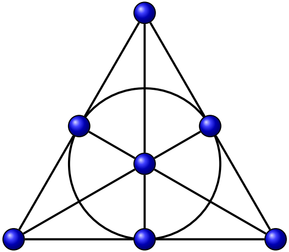
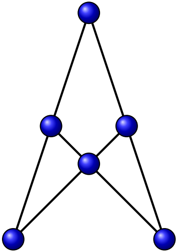


Figure: Partial linear spaces don't tolerate this kind of thing.

A couple of partial linear spaces





Flag-transitive linear spaces

- Come in two flavours – affine and almost simple.
- Classification of flag-transitive linear spaces announced in 1990 (Buekenhout et al),
  - Completed in 2002,
  - 1-dimensional affine case as yet unresolved.
- Classification relied extensively on the O'Nan-Scott Theorem and the classifications of finite simple groups and finite rank 2 groups.

Rank 3 partial linear spaces

The most symmetric partial linear spaces which aren't linear spaces have rank 3, where the three orbits are:

1. pairs of equal points;
2. pairs of collinear points;
3. pairs of non-collinear points.

Primitive rank 3 partial linear spaces

- Devillers classified the finite flag-transitive partial linear spaces with a primitive rank 3 automorphism group of almost simple type in 2005.
- She then classified the ones with an automorphism group of grid type in 2008.
- Flag-transitive partial linear spaces with an affine primitive rank 3 automorphism group are currently being worked on.

Imprimitive rank 3 partial linear spaces?

- I am currently working on classifying the partial linear spaces with a quasiprimitive imprimitive rank 3 automorphism group.
- I will also work on the ones with an innately transitive rank 3 automorphism group, as soon as I'm done classifying the groups.
- I'm just using finite geometry to justify more group theory.

Why would I do this?

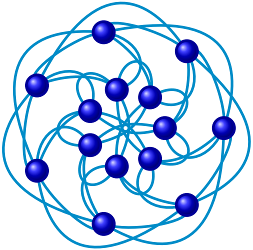
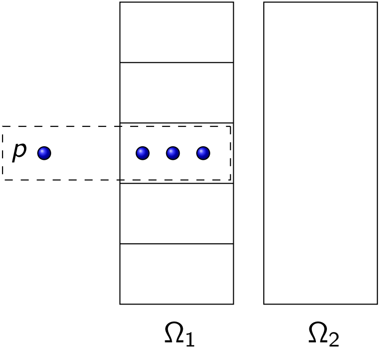


Figure: The BD flower, with automorphism group $\mathrm{PSL}(3,2) \times C_2$ – that's why.

Start with G, a rank 3 group



1. Three orbits of G_p on the set;
2. One of the non-trivial orbits, Ω_1, forms the set of points collinear to p;
   - The other orbit, Ω_2, forms the non-collinear points;
3. Find blocks of imprimitivity of G_p on Ω_1;
4. Points in the block together with p form a line in a geometry;
5. Check if the resulting geometry is a PLS.

Here is the Petersen graph for your viewing pleasure