

Some Recent Results on the Covering Numbers of Symmetric Groups, and Sporadic Groups

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Abstract

- We say that a group G has a finite covering if G is a set theoretical union of finitely many proper subgroups. The minimal number of subgroups needed for such a covering is called the covering number of G denoted by $\sigma(G)$.
- Let S_n be the symmetric group on n letters. For odd n Maroti determined $\sigma(S_n) = 2^{n-1}$ with the exception of $n = 9$, and gave estimates for n even showing that $\sigma(S_n) \leq 2^{n-2}$. Using *GAP* calculations, as well as incidence matrices and linear programming, we show that $\sigma(S_8) = 64$, $\sigma(S_{10}) = 221$, $\sigma(S_{12}) = 761$. We also show that Maroti's result for odd n holds without exception proving that $\sigma(S_9) = 256$.
- We establish in addition that $\sigma(A_9) = 157$, $\sigma(A_{11}) = 2751$, the Mathieu group M_{12} has covering number 208, and improve the estimate for the Janko group J_1 given by P.E. Holmes.

Introduction

- It is well known that no group is the union of 2 proper subgroups.
- The question which integers can be covering numbers of groups was raised by Tomkinson.
- It was proved so far that there are no groups with covering number 2, 7, 11, 19, 21, 22, 25 (Tomkinson, Detoni, Lucchini,...), and the smallest integer for which it is not known whether it is a covering number for a group is **$n = 27$** .

The Covering Number

- **Theorem 1 (Tomkinson, 1997):** *Let G be a finite soluble group and let p^α be the order of the smallest chief factor having more than one complement. Then $\sigma(G) = p^\alpha + 1$.*
- The author suggested the investigation of the covering number of simple groups.

Linear Groups

- **Theorem 2 (Bryce, Fedri, Serena, 1999)**
- $\sigma(G) = 1/2 q(q+1)$ when q is even,
- $\sigma(G) = 1/2 q(q+1) + 1$ when q is odd,

where $G = \text{PSL}(2, q)$, $\text{PGL}(2, q)$, or $\text{GL}(2, q)$,
and $q \neq 2, 5, 7, 9$.

Suzuki Groups

- **Theorem 3 (Lucido, 2001)**
- $\sigma(\text{Sz}(q)) = \frac{1}{2} q^2(q^2+1)$,
where $q = 2^{2m+1}$.

Sporadic Simple Groups

Theorem 6 (P.E. Holmes, 2006)

$$\sigma(m_{11})=23, \sigma(m_{22})= 771, \sigma(m_{23})=41079,$$

$$\sigma(Ly) = 112845655268156,$$

$$\sigma(O'N) = 36450855$$

$$5165 \leq \sigma(J_1) \leq 5415$$

$$24541 \leq \sigma(M^C L) \leq 24553.$$

The author has used *GAP*, the *ATLAS*, and Graph Theory.

Symmetric and Alternating Groups

- **Theorem 4** (Maroti, 2005)
- $\sigma(S_n) = 2^{n-1}$ if n is odd, $n \neq 9$
- $\sigma(S_n) \leq 2^{n-2}$ if n is even.
- $\sigma(A_n) \geq 2^{n-2}$ if $n \neq 7, 9$, and $\sigma(A_n) = 2^{n-2}$ if n is even but not divisible by 4.
- $\sigma(A_7) \leq 31$, and $\sigma(A_9) \geq 80$.

Alternating Groups

- **Theorem 5 (Luise-Charlotte Kappe, Joanne Redden, 2009)**
- $\sigma(A_7) = 31$
- $\sigma(A_8) = 71$
- $127 \leq \sigma(A_9) \leq 157$
- $\sigma(A_{10}) = 256.$

Recent results

- We can now prove the exact numbers:
- $\sigma(S8) = 64$.
- $\sigma(S9) = 256$.
- $\sigma(S10) = 221$.
- $\sigma(S12) = 761$.
- $\sigma(A9) = 157$.
- $\sigma(A11) = 2751$.
- $5316 \leq \sigma(SJ1) \leq 5413$.

- It is sufficient to consider the number of maximal subgroups of G needed to cover all maximal cyclic subgroups of G .
- We used *GAP* for the distribution of the elements in the maximal subgroups
- We estimated the limits by a *Greedy Algorithm*.

- **Easy case:** When the elements are partitioned into the subgroups of a conjugacy class
- **Harder case:** When the elements of a certain cyclic structure are not partitioned.
- **Approaches:**
 - Greedy algorithm
 - Incidence matrices and Combinatorics
 - Linear programming

S7

Maximal subgroups	Order of Class Representative	Size
MS1 = A_7	2520	1
MS2 = S_6	720	7
MS3 = S_3 x S_4	144	35
MS4 = C_2 x S_5	240	21
MS5 = (C_7:C_3):C_2	42	120

Distribution of the Elements of S7:

Order	Cyclic Structure	Size	MS1=A_7	MS2	MS3	MS4	MS5
1	1	1	1				
2	(12)	21	0	15	9	11	0
2	(12)(34)		X				
2	(12)(34)(56)	105	0	15,P	9	15	7
3	(123)		X				
3	(123)(456)		X				
4	(1234)	210	0	90	6,P	30	0
4	(1234)(56)		X				
5	(12345)		X				
6	(123456)	840	0	120,P	0	0	14
6	(123)(45)	420	0	120	36	40	0
6	(123)(45)(67)		X				
7	(1234567)	720	X				
10	(12345)(67)	504	0	0	0	24,P	0
12	(1234)(567)	420	0	0	12,P	0	0

S7

- It is clear from the table why $\sigma = 2^{7-1}$
- The group is covered by A_7 (MS1), the 7 groups S_6 in MS2, the 35 groups in MS3, and the 21 groups in MS4: $1+7+35+21=64=2^6$.
- $\sigma(S7) = 64$.

S8

Maximal subgroups	Order of Class Representative	Size
MS1 = A_8	20160	1
MS2 = S_3 x S_5	720	56
MS3 = C_2 x S_6	1440	28
MS4 = S_7	5040	8
MS5=(((C_2xD_8):C_2):C_3):C_2):C_2	384	105
MS6 = (S_4 x S_4): C_2	1152	35
MS7 = PSL(3,2): C_2	336	120

S8

Distribution of Elements:

Order	Cyclic Structure	Size	MS1	MS2	MS3	MS4	MS5	MS6	MS7
1	1	1	1	1	1	1	1	1	1
2	2 ¹	28	0	13(26)	16(16)	21(6)	4(15)	12(15)	0
2	2 ²	210	210, P	45(12)	60(8)	105(4)	18(9)	42(7)	0
2	2 ³	420	0	45(6)	60(4)	105(2)	28(7)	36(3)	28(8)
2	2 ⁴	105	105, P	0	15(4)	0	25(25)	33(11)	21(24)
3	3 ¹	112	112, P	22(11)	40(10)	70(5)	0	16(5)	0
4	2x4	2520	2520,P	90(2)	180(2)	630(2)	24,P	72,P	0
4	4 ¹	420	0	30(4)	90(6)	210(4)	12(3)	12,P	0
4	2 ² x 4	1260	0	0	90(2)	0	36(3)	180(5)	0
4	4 ²	1260	1260,P	0	0	0	60(5)	108(3)	42(4)
5	5	1344	1344,P	24,P	144(3)	504(3)	0	0	0
6	2x3	1120	0	100(5)	160(4)	420(3)	0	96(3)	0
6	2x2x3	1680	1680,P	90(3)	120(2)	210,P	0	48,P	0
6	2x3 ²	1120	0	40(2)	40,P	0	32(3)	0	0
6	6	3360	0	0	120,P	840(2)	32,P	0	56(2)
6	2 x 6	3360	3360,P	0	120,P	0	32,P	192(2)	0
7	7	5760	5760,P	0	0	720,P	0	0	48,P
8	8	5040	0	0	0	0	48,P	144,P	84(2)
10	2 x 5	4032	0	72,P	144,P	504,P	0	0	0
12	3 x 4	3360	0	60,P	0	420,P	0	96,P	0
15	3 x 5	2688	2688,P	48,P	0	0	0	0	0

S8

- Here are the maximal subgroups and the distribution of the elements of S_8 in the representatives of the maximal subgroups. In parentheses the small numbers mean in how many representatives each element is to be found.
- Example: Each element of order 6 of type 2×3 i.e. $(1,2)(3,4,5)$ is to be found in 3 representatives of MS4, and in each representative of MS4 there are 420 such elements.
- The group is covered by A_8 (MS1), the 28 groups in MS3, and the 35 groups in MS6, i.e. $1+28+35 = 64 = 2^6$.
- $\sigma(S_8) = 64$.
- The difficulty consists to prove that this is a minimal covering indeed. We do that computationally using *GAP* and *Gurobi*.

The Covering Number of $S_{\downarrow 10}$

- To determine a minimal covering by maximal subgroups, it suffices to find a minimal covering of the conjugacy classes of maximal cyclic subgroups by maximal subgroups of the group.

Maximal subgroups

Maximal subgroups (3977)	Order of Class Representative	Size
MS1 = A_10	1814400	1
MS2=S_4 x S_6	17280	210
MS3 = S_3 x S_7	30240	120
MS4 = C_2 x S_8	80640	45
MS5 = S_9	362880	10
MS6= C_2 x (((C_2xC_2xC_2xC_2):A_5):C_2	3840	945
MS7 = (S_5 x S_5):C_2	28800	126
MS8 = (A_6.C_2):C_2	1440	2520

Distribution of elements generating maximal cyclic subgroups:

Order	Cyclic Structure	Size	MS1	MS2	MS3	MS4	MS5	MS6	MS7	MS8
ODD										
4	2 ² x 4	56700	0	1080 ₄	1890 ₄	3780 ₃	11340 ₂	180 ₃	900 ₂	0
4	2x4 ²	56700	0	540 ₂	0	1260,P	0	300 ₅	1800 ₄	90 ₄
6	2 ³ x3	25200	0	480 ₄	840 ₄	1680 ₃	2520, P	0	600 ₃	0
6	2x3 ²	50400	0	1200 ₅	1680 ₄	2240 ₂	10080 ₂	160 ₃	800 ₂	0
6	2 ² x6	75600	0	360,P	0	3360 ₂	0	240 ₃	2400 ₄	0
6	3x6	201600	0	960,P	1680,P	0	20160,P	0	0	240 ₃
8	8	226800	0	0	0	5040,P	45360	240,P	0	180
10	10	362880	0	0	0	0	0	384,P	2880,P	144,P
12	3²4	50400	0	240,P	840₂	0	0	160₃	0	0
14	2x7	259200	0	0	2160,P	5760,P	25920,P	0	0	0
20	4x5	181440	0	964,P	0	0	18144,P	0	1440,P	0
30	2x3x5	120960	0	0	1008,P	2688,P	0	0	960,P	0
-EVEN-----	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----
6	2 x 6	151200	P	720, P	2520 ₂	6720 ₂	30240 ₂	160 P	0	0
9	9	403200	P	0	0	0	40320, P	0	0	0
12	4x6	151200	P	720, P	0	0	0	160, P	2400 _{x2}	0
12	2x3x4	151200	P	1440 ₂	2520 ₂	3360 P	15120 P	0	1200 P	0
21	3 x 7	172800	P	0	1440 P	0	0	0	0	0
8	8x2	226800	P	0	0	5040,P	0	240,P	3600 ₂	180 ₂

Theorem 1:
The Covering Number of S_{10} is 221.

- *Sketch of the Proof:*
- It is not difficult to see from the Inventory that the groups from MS3, MS5, and MS7 represent a covering of the odd permutations, and MS1={A10} covers the even.
- We want to minimize this covering.
- The problematic elements are of structure $3 \times 3 \times 4$, of order 12.
- The proof further involves *Incidence matrices*, and *Combinatorics*.

S10

- We first found that the Covering number has **upper bound**:
 $MS1+MS3+MS5+MS7 = 1+120+10+126=257$.
- However, we ran a Greedy algorithm on MS3 and found out that 84 groups only from MS3 are sufficient to cover the elements of type $3^2 \times 4$. So:
- $\sigma \leq 1+84+10+126=221$.
- The upper bound was reduced.
-
- **The lower bound:** The elements of type $3^2 \times 4$ are 50400. If they were partitioned in MS3 we would have needed $50400/840 = 60$.
- So, we need at least 61 from them.
- $1+61+10+126= 198$.
-
- **Hence $198 \leq \sigma \leq 221$.**

Incidence matrices

- Let V , and U are two collections of objects. Call the objects in V elements, and the objects in U sets.
- The incidence structure between U and V can be represented by the incidence matrix $A(a_{ij})$ of (V, U) :
- $a_{ij} = \begin{cases} 1 & \text{if } v_i \in U_j \\ 0 & \text{if } v_i \notin U_j \end{cases}$
- Let W be a sub-collection of U . We define a vector $x(W) = (x_1, x_2, \dots, x_{|U|})^T$ as follows
- $x_j = \begin{cases} 1 & \text{if } u_j \in W \\ 0 & \text{if } u_j \notin W \end{cases}$
- Let $A^* x(W) = y(W) = (y_1, y_2, \dots, y_{|V|})^T$, where $y_i \geq 0$.
- If $y_i = 0$, then $v_i \notin U \cup W$, and

The elements of type $3^3 \cdot 4$

- There are 50,400 elements of type $3^3 \cdot 4$ in S_{10} . They are to be found in MS3, but are not partitioned.
- Each class of MS3 contains 840 such elements, and each element is in exactly 2 subgroups of MS3.
- **Because the subgroups of MS3 are isomorphic to $S_3 \times S_7$, we can label them by the letters fixed by the respective S_7 , i.e.**
- **$MS3 = \{H(k_1, k_2, k_3), k_1, k_2, k_3 \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}, k_1 < k_2 < k_3\}$.**
- So, our incidence matrix will contain 120 columns, labeled by the members of MS3.
- The rows are the maximal cyclic subgroups generated by our elements.

There are 6 cyclic subgroups of order 12 in the intersection of $H(i \downarrow 1, i \downarrow 2, i \downarrow 3)$ and $H(i \downarrow 4, i \downarrow 5, i \downarrow 6)$ and each one of them contains 4 elements of type $3^3 \cdot 4$: thus the 50,400 elements of type $3^3 \cdot 4$ are partitioned into $50,400/4=12,600$ equivalence classes. Our incidence matrix will have 2100 rows.

Confirming the result of the Greedy algorithm

- We have an incidence (0 -1) matrix A of size 2100 x 120 with exactly 2 entries equal to 1 in each row.
- If $x(U) = (1, 1, \dots, 1) \uparrow T$, $y(W) = A * x(U) = (2, 2, \dots, 2) \uparrow T$.
- We want to determine the maximum numbers of 0-s entries contained in a $x(W)$ vector, so that the $y(W)$ vector has all non-zero entries.
- **We can achieve that by removing the maximal subset $\{u \downarrow 1, u \downarrow 1, \dots, u \downarrow 1\}$ of U with pairwise non-trivial intersection.**

Combinatorics

- **THEOREM (Erdos, Ko, Rado):** The maximal number m of k -subsets A_1, A_2, \dots, A_m of an n -set S that are pairwise non-disjoint is $m \leq \binom{n-1}{k-1}$.
- The upper bound is best possible, and it is attained when A_i are precisely those k -subsets of S which contain a chosen fixed element of S .

Corollary

Proposition:

The elements of type $3^3 \cdot 4$ in S_{10} are covered by 84 groups from MS_3 , and this is a minimal covering. In particular,

$M = MS_3 - D$, where

$$D = \{ H(0, k \downarrow 1, k \downarrow 2); k \downarrow 1, k \downarrow 2 \in \{1, 2, \dots, 9\}, k \downarrow 2 < k \downarrow 3 \}$$

is a minimal covering.

Proof:

According to the Theorem ($n=10, k=3$), the maximal subset $\{u \downarrow 1, u \downarrow 2, \dots, u \downarrow t\}$ of U with pairwise non-trivial intersection has cardinality: $m = \binom{9}{2} = 36$. Therefore,

- $120 - 36 = 84$.

Proof of Theorem 1

- We shall see that $\sigma(S_{10}) = |\text{MS1}| + |\text{MS5}| + |\text{MS7}| + 84 = 221$.
- The elements of order 21 are only to be found in MS1 and MS3, in both they are partitioned, so we take $\text{MS1} = \{A_{10}\}$, size 1.
- The elements of order 10 are partitioned in MS6, MS7, and MS8. MS7 has the least size: 126.
- The elements of order 14, type $2 \cdot 7$ are partitioned in MS3, and MS5. If $H(0, k_1, k_2)$ is removed from MS3, they will no longer be covered by MS3. They can only be covered by all 10 members of MS5.
- Together with the result for the elements of type $3 \cdot 3 \cdot 4$, we have:
- $\sigma(S_{10}) = 1 + 126 + 10 + 84 = 221$.

S9

Maximal subgroups (1376)	Order of Class Representative	Size
MS1 = A_9	181440	1
MS2 = S_4 x S_5	2880	126
MS3 = S_3 x S_6	4320	84
MS4 = C_2 x S_7	10080	36
MS5 = S_8	40320	9
MS6 = (((C_3x((C_3xC_3):C_2)):C_2):C_3):C_2	1296	280
MS7 = (((C_3xC_3):Q_8):C_3):C_2	432	840

S9

Distribution of Elements:

Order	Cyclic Structure	Size	MS1	MS2	MS3	MS4	MS5	MS6	MS7
1	1	1	1	1	1	1	1	1	1
2	2 ¹		0						
2	2 ²								
2	2 ³		0						
2	2 ⁴								
3	3 ¹								
3	3 ²								
3	3 ³								
4	2x4	7560	7560,P						
4	4 ¹	756	0	36(6)	90(10)	210(10)	420(5)	0	0
4	2 ² x 4	11340	0	180(2)	270(2)	630(2)	1260,P	162(4)	0
4	4 ²								
5	5	3024	3024,P						
6	2x3	2520	0	220(11)	270(9)	490(7)	1120(4)	36(4)	0
6	2 ² x3	7560	7560,P						
6	2x3 ²	10080	0	160(2)	360(3)	280,P	1120,P	36, P	0
6	6	10080	0	0	120,P	840(3)	3360(3)	36, P	56(2)
6	2 x 6	30240	30240,P						
6	2 ³ 3	2520	0	60(3)	30, P	210(3)	0	36(4)	0
6	3x6	20160	0	0	240,P	0	0	288(4)	72(3)
7	7	25920	25920,P						
8	8	45360	0	0	0	0	5040,P	0	108(2)
9	9	40320	40320,P						
10	2 x 5	18144	0	144,P	432(2)	1008(2)	4032(2)	0	0
10	2 ² 5	9072	9072,P						
12	3 x 4	15120	0	360	180,P	420,P	3360	0	0
14	2x7	25920	0	0	0	720,P	0	0	0
15	3 x 5	24192	24192,P						
20	4x5	18144	0	144,P	0	0	0	0	0

S9

- Here is the distribution of the elements of S_9 in the representatives of the maximal subgroups. Here is how the lower and the upper bound are clearly to be seen:
- We definitely need:
- MS1=A_9 (1 group)
- MS2 (126 groups) to cover the elements of order 20.
- MS4 (36 groups) to cover the elements of order 14, and 12.
- MS5 (9 groups) to cover the elements of order 8, and $((1,2,3)(4,5,6)(7,8))$.
- Then, if you take all the 84 groups of MS3, we'll cover 3 types of elements of order 6. So, 84 more groups add up to **256: the upper bound**.
- **The lower bound.:**
- If we cover the elements of type 3x6 (20160) by groups from MS6 instead (where they are not partitioned), we would have needed at least $20160/288=70$ groups. So, **$1+126+36+9+71=243 \geq \sigma$**
- **Hence, $243 \leq \sigma \leq 256$.**

Linear Programming

- **Theorem:** The covering number $\epsilon(S_9) = 256$.
- **Proposition:** The elements 3×6 have a minimal covering by 84 subgroups.
- **Proof:** *GAP and Gurobi.*
- *GAP* sets a system of linear equations corresponding to covering
- *Gurobi* (optimizer of a system of linear equations) finds the minimal covering.
- Same approach was used for the Mathieu group M_{12} and the Janko group J_1 .

J1 and KoKo

- Practically, the paper was ready by August 2014. We announced it at: <http://arxiv.org/abs/1409.2292>
- However, we wanted to achieve the best possible (the smallest) range for J1 on more powerful machines. Which we did on the new super computer KoKo installed by Max Planck at the FAU Harbor branch. Here below are some characteristics of KoKo:
- **400 Intel Xeon Cores; 1000's of Intel Xeon Phi Cores; 128GB of RAM per node; Scientific Linux 6.5; 160 terabites memory.** For more details see: <http://hpc.fau.edu>

J1 and KoKo_2

- The program for the elements of order 11 finished in about 2 1/2 hour. It took 196 subgroups to cover the elements of order 11 in J1.
- However, although powerful parallel computing was done on the super-computer, with optimal parameters, and using 8 nodes, it took a while to get to:
- 41146230 38175957 99% 0% 0% 751.00000 650.36569 13.4% 618 581650s (Jan 23);
253860990 231372205 99% 0% 0% 752.00000 653.04421 13.2% 476 2244640s (March 10).
- **Interpretation:** The lower bound we got for the elements of order 7 is 654, the upper bound – 751, and the discrepancy between the two numbers is 13.2%.
- It took such a long time, because we deal with a difficult system of equations to optimize. Another reason is that GUROBI optimizer does not have check points, so every time KoKo was stopped for maintenance, we had to restart the program. The last calculation from March 10 took 476 2244640s = 26.3 days...
- With the newest results, we can claim now that the covering number for J1 is between:
- 1540+2926+196+654=5316 and
- 1540+2926+196+751=5413, i.e.:
- **$5316 \leq \sigma(J_1) \leq 5413$**
- We also tried MINION, but the problem has no better solution than the one GUROBI provided. This is as far as we can push the bounds for J1 with current techniques.

Thank you!

QUESTIONS?