A Tiling and $(0, 1)$-Matrix Existence Problem

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A $(0,1)$-Matrix and Tiling Problem

Let $R = (r_1, r_2, \ldots, r_m, r_{m+1} = 0)$ and $S = (s_1, s_2, \ldots, s_n)$ be nonnegative integral vectors.

A $(0,1)$-Matrix and Tiling Problem

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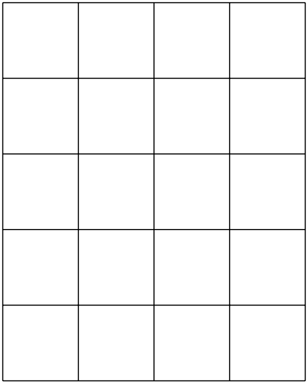
Question: Can a $(m+1) \times n$ checkerboard be tiled with vertical dimers and monomers so that there are r_i dimers with the upper half of the dimer in row i and s_i dimers in column i?

A $(0,1)$-Matrix and Tiling Problem

Example: $R = (2,2,1,2,0)$; $S = (2,1,2,2)$



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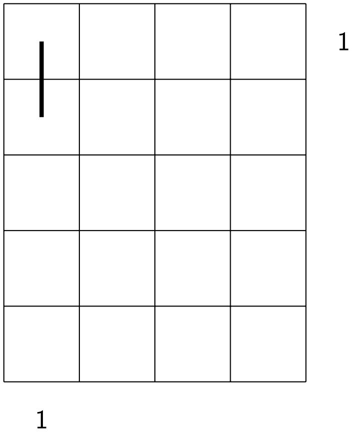

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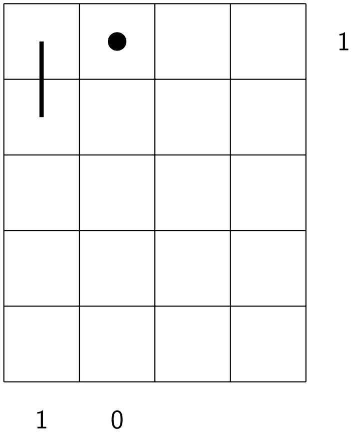

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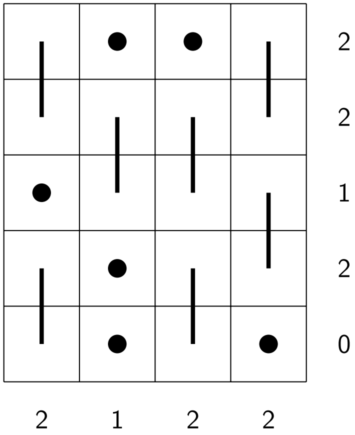

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2. The existence of a $(0,1)$-matrix where no consecutive 1's occur in a column.

3. Phrase it as a linear programming problem and look for a $0,1$ solution. ($a_{11} + a_{12} + \cdots + a_{1n} = r_1$, etc.)

Our Point of View

The existence of a $(0, 1)$-matrix where no consecutive 1's occur in a column.

Definition

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Let $A(R, S)$ be the set of all $(0, 1)$-matrices with

- row sum vector R
- column sum vector S.

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-Studied by H.J. Ryser, D. Gale, D.R. Fulkerson, R.M Haber, and R. Brualdi.

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Let $A_1(R, S)$ be the set of all $(0, 1)$-matrices with

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Question

When is $A_1(R, S)$ nonempty?

Example

$R = (1, 1, 3, 2, 2, 3)$; $S = (3, 1, 3, 1, 1, 3)$

						1
						1
						3
						2
						2
						3
3	1	3	1	1	3	

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1						1
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1						1
		1				1
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1					X	1
		1			X	1
						3
						2
						2
						3
3	1	3	1	1	3	

An Observation

Observation: If $M \in A_1(R, S)$ then we can entry wise sum rows r_i and r_{i+1} and get a matrix in $A((r_1, \dots, r_{i-1}, r_i + r_{i+1}, r_{i+2}, \dots), S)$.

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The Gale-Ryser Theorem characterizes when $A(R, S)$ is nonempty.

Definition

Majorization:

- Nonincreasing integral vectors: $a = (a_1, a_2, \ldots a_n)$ and $b = (b_1, b_2, ...b_m)$. -append zeros to make them equal length (say $n \geq m$).
- a is majorized by b, denoted $a \preceq b$ when

$$a_1 + a_2 + \cdots + a_k \leq b_1 + b_2 + \ldots b_k \text{ for all } k$$

and

$$a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n$$

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Example

Example: $(3, 2, 1, 0) \preceq (3, 3, 0, 0)$ since

$3 \leq 3$

$3 + 2 \leq 3 + 3$

$3 + 2 + 1 = 3 + 3 + 0$

$3 + 2 + 1 + 0 = 3 + 3 + 0 + 0$

Definition

Conjugate of a nonnegative integral vector:

$$R = (3, 2, 3, 1) \qquad\qquad R^* = (4, 3, 2, 0, \ldots, 0)$$

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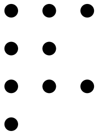

Existence Theorem for $A(R, S)$

Theorem (Gale-Ryser, 1957)

If $R = (r_1, r_2, \ldots, r_m)$ and $S = (s_1, s_2, \ldots, s_n)$ are nonnegative integral vectors such that S is nonincreasing, then there exists an $m \times n, (0, 1)$-matrix with row sum vector R and column sum vector S if and only if $S \preceq R^*$.

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For $A_1(R, S)$,

$S \preceq (r_1 + r_2, r_3 + r_4, r_5)^*$
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Definition

Let $Q_R = \{(r_1 + r_2, r_3 + r_4, r_5), (r_1 + r_2, r_3, r_4 + r_5), (r_1, r_2 + r_3, r_4 + r_5)\}$.

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Observation: If $A_1(R, S)$ is nonempty then $S \preceq q^*$ for all $q \in Q_R$.

Is this condition sufficient to show $A_1(R, S)$ is nonempty?

Existence Theorem for $A_1(R, S)$

Theorem (N., Shader)

If $R = (r_1, r_2, \ldots, r_m)$ and $S = (s_1, s_2, \ldots, s_n)$ are nonnegative integral vectors such that S is nonincreasing, then there exists an $m \times n$ $(0,1)$-matrix with no two 1's occurring consecutively in a column and with row sum vector R and column sum vector S if and only if

$$S \preceq q^* \quad \forall q \in Q_R.$$

Proofs of the Gale-Ryser Theorem

-direct combinatorial arguments
-network flows

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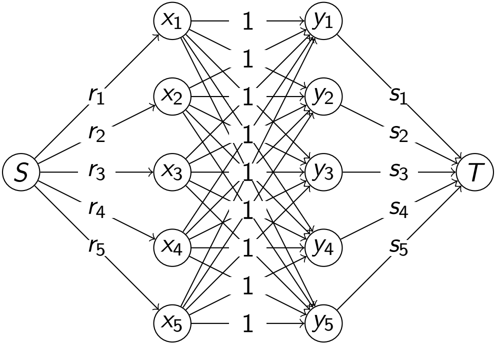
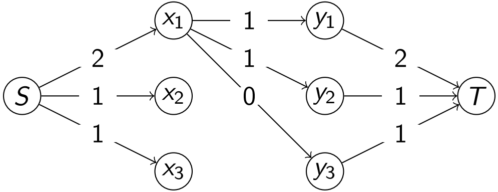
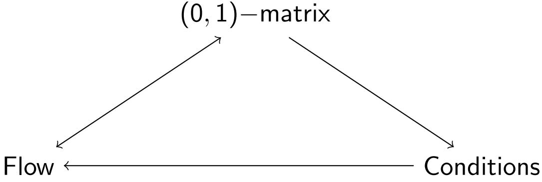


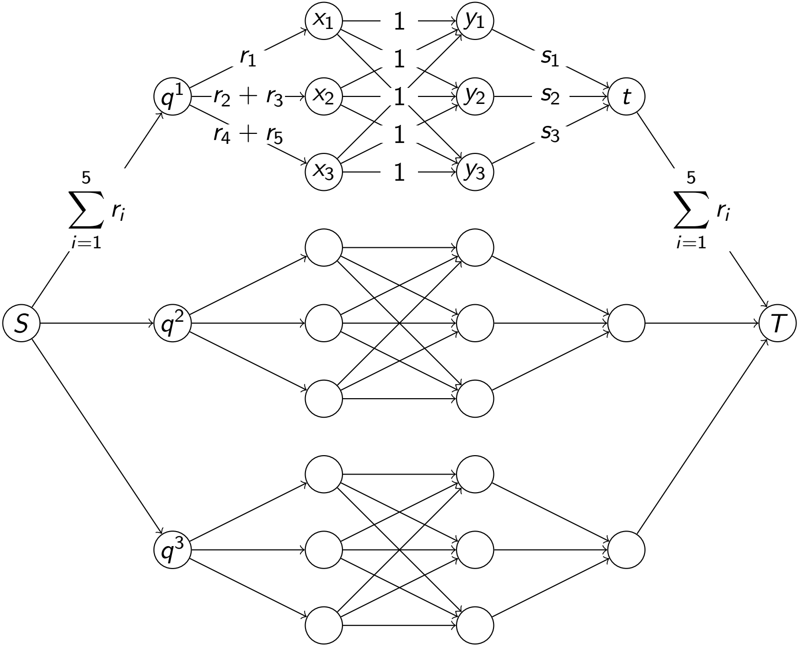
Network Flow for $A(R, S)$



$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

For $A(R, S)$





x_1
x_2
x_3
y_1
y_2
y_3
1
1
1
1
1
r_1
$r_2 + r_3$
$r_4 + r_5$
s_1
s_2
s_3
q^1
q^2
q^3
t
S
T
$\sum_{i=1}^{5} r_i$
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  - Let M be a (0, 1)-matrix in $A_1((r_1, r_2, \ldots, r_i - 1, \ldots, r_m), (s_1, \ldots, s_n - 1))$.

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  - If there is a 1 in the (i, n) position, argue that with a switch this can be changed to a 0.
  - Put a 1 in the (i, n) position and use switches to remove any consecutive 1's.
- This completes the induction on the number of 1's in the last column and in turn the induction on the number of columns.

□

Graph of $A_1(R, S)$

Definition

The *graph of* $A_1(R, S)$ is an undirected graph with:

- vertices are the matrices in $A_1(R, S)$
- $M_1 \sim M_2$ if and only if the matrix M_1 can be changed to M_2 with one basic switch.

Further Work

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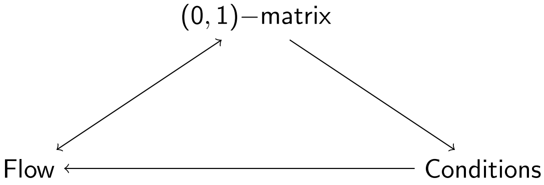
- What if every 1 is followed by j zeros: $A_j(R, S)$?
- Further study of the network flow connection.

Further Work

$A(R,S)$:



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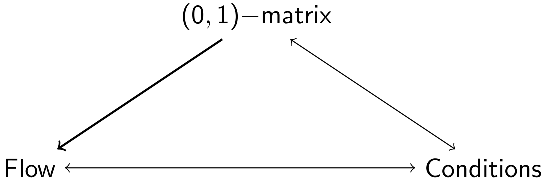

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