Gröbner bases methods in affine and projective variety codes

Cícero Carvalho
Faculdade de Matemática
Universidade Federal de Uberlândia

CoCoA 2015 - Combinatorics and Computer Algebra
Colorado State University, July 19–25, 2015

Partially supported by CNPq (480477/2013-2)

Facts from Gröbner bases theory

Let $\mathcal{M}$ be the set of monomials of $k[X_1, \ldots, X_n] =: k[\mathbf{X}]$ and endow $\mathcal{M}$ with a monomial order $\prec$. Given $f \in k[\mathbf{X}] \setminus \{0\}$ the leading monomial of f ($\text{lm}(f)$) is the greatest monomial appearing in f. Let $I \subset k[\mathbf{X}]$ be an ideal, we say that $\{g_1, \ldots, g_s\} \subset I$ is a Gröbner basis for I (w.r.t. $\prec$) if the leading monomial of any nonzero polynomial in I is multiple of $\text{lm}(g_i)$ for some $i = 1, \ldots, s$. One can prove that such a set is a basis for I in the usual sense $I = (g_1, \ldots, g_s)$ and that any ideal has a Gröbner basis.
A related and very important concept is the *footprint* of I (w.r.t. $\prec$) defined as

$\Delta(I) = \{M \in \mathcal{M} \mid M \text{ is not the leading monomial of any polynomial in } I\}$

Buchberger (1965) proved that $\{M + I \mid M \in \Delta(I)\}$ is a basis for $k[\mathbf{X}]/I$ (considered as a k-vector space).
If I is homogeneous then $\{M + I \mid M \in \Delta(I), \deg(M) = d\}$ is a basis for $k[\mathbf{X}]_d/I(d)$.
We also have that $M \in \Delta(I)$ if and only if M is not a multiple of $\text{lm}(g_i)$ for all $i = 1, \ldots, s$.

Facts from Gröbner bases theory

Let $\mathcal{M}$ be the set of monomials of $k[X_1, \ldots, X_n] =: k[\mathbf{X}]$ and endow $\mathcal{M}$ with a monomial order $\prec$. Given $f \in k[\mathbf{X}] \setminus \{0\}$ the leading monomial of f ($\mathrm{lm}(f)$) is the greatest monomial appearing in f. Let $I \subset k[\mathbf{X}]$ be an ideal, we say that $\{g_1, \ldots, g_s\} \subset I$ is a Gröbner basis for I (w.r.t. $\prec$) if the leading monomial of any nonzero polynomial in I is multiple of $\mathrm{lm}(g_i)$ for some $i = 1, \ldots, s$. One can prove that such a set is a basis for I in the usual sense $I = (g_1, \ldots, g_s)$ and that any ideal has a Gröbner basis.
A related and very important concept is the *footprint* of I (w.r.t. $\prec$) defined as

$\Delta(I) = \{M \in \mathcal{M} \mid M \text{ is not the leading monomial of any polynomial in } I\}$

Buchberger (1965) proved that $\{M + I \mid M \in \Delta(I)\}$ is a basis for $k[\mathbf{X}]/I$ (considered as a k-vector space).
If I is homogeneous then $\{M + I \mid M \in \Delta(I), \deg(M) = d\}$ is a basis for $k[\mathbf{X}]_d/I(d)$.
We also have that $M \in \Delta(I)$ if and only if M is not a multiple of $\mathrm{lm}(g_i)$ for all $i = 1, \ldots, s$.

Facts from Gröbner bases theory

Let $\mathcal{M}$ be the set of monomials of $k[X_1, \ldots, X_n] =: k[\mathbf{X}]$ and endow $\mathcal{M}$ with a monomial order $\prec$. Given $f \in k[\mathbf{X}] \setminus \{0\}$ the leading monomial of f ($\mathrm{lm}(f)$) is the greatest monomial appearing in f. Let $I \subset k[\mathbf{X}]$ be an ideal, we say that $\{g_1, \ldots, g_s\} \subset I$ is a Gröbner basis for I (w.r.t. $\prec$) if the leading monomial of any nonzero polynomial in I is multiple of $\mathrm{lm}(g_i)$ for some $i = 1, \ldots, s$. One can prove that such a set is a basis for I in the usual sense $I = (g_1, \ldots, g_s)$ and that any ideal has a Gröbner basis.
A related and very important concept is the *footprint* of I (w.r.t. $\prec$) defined as
$\Delta(I) = \{M \in \mathcal{M} \mid M \text{ is not the leading monomial of any polynomial in } I\}$
Buchberger (1965) proved that $\{M + I \mid M \in \Delta(I)\}$ is a basis for $k[\mathbf{X}]/I$ (considered as a k-vector space).
If I is homogeneous then $\{M + I \mid M \in \Delta(I), \deg(M) = d\}$ is a basis for $k[\mathbf{X}]_d/I(d)$.
We also have that $M \in \Delta(I)$ if and only if M is not a multiple of $\mathrm{lm}(g_i)$ for all $i = 1, \ldots, s$.

Facts from Gröbner bases theory

Let $\mathcal{M}$ be the set of monomials of $k[X_1, \ldots, X_n] =: k[\mathbf{X}]$ and endow $\mathcal{M}$ with a monomial order $\prec$. Given $f \in k[\mathbf{X}] \setminus \{0\}$ the leading monomial of f ($\text{lm}(f)$) is the greatest monomial appearing in f. Let $I \subset k[\mathbf{X}]$ be an ideal, we say that $\{g_1, \ldots, g_s\} \subset I$ is a Gröbner basis for I (w.r.t. $\prec$) if the leading monomial of any nonzero polynomial in I is multiple of $\text{lm}(g_i)$ for some $i = 1, \ldots, s$. One can prove that such a set is a basis for I in the usual sense $I = (g_1, \ldots, g_s)$ and that any ideal has a Gröbner basis.
A related and very important concept is the *footprint* of I (w.r.t. $\prec$) defined as
$\Delta(I) = \{M \in \mathcal{M} \mid M \text{ is not the leading monomial of any polynomial in } I\}$
Buchberger (1965) proved that $\{M + I \mid M \in \Delta(I)\}$ is a basis for $k[\mathbf{X}]/I$ (considered as a k-vector space).
If I is homogeneous then $\{M + I \mid M \in \Delta(I), \deg(M) = d\}$ is a basis for $k[\mathbf{X}]_d/I(d)$.
We also have that $M \in \Delta(I)$ if and only if M is not a multiple of $\text{lm}(g_i)$ for all $i = 1, \ldots, s$.

Facts from Gröbner bases theory

Let $\mathcal{M}$ be the set of monomials of $k[X_1, \ldots, X_n] =: k[\mathbf{X}]$ and endow $\mathcal{M}$ with a monomial order $\prec$. Given $f \in k[\mathbf{X}] \setminus \{0\}$ the leading monomial of f ($\mathrm{lm}(f)$) is the greatest monomial appearing in f. Let $I \subset k[\mathbf{X}]$ be an ideal, we say that $\{g_1, \ldots, g_s\} \subset I$ is a Gröbner basis for I (w.r.t. $\prec$) if the leading monomial of any nonzero polynomial in I is multiple of $\mathrm{lm}(g_i)$ for some $i = 1, \ldots, s$. One can prove that such a set is a basis for I in the usual sense $I = (g_1, \ldots, g_s)$ and that any ideal has a Gröbner basis.
A related and very important concept is the *footprint* of I (w.r.t. $\prec$) defined as

$\Delta(I) = \{M \in \mathcal{M} \mid M$ is not the leading monomial of any polynomial in $I\}$

Buchberger (1965) proved that $\{M + I \mid M \in \Delta(I)\}$ is a basis for $k[\mathbf{X}]/I$ (considered as a k-vector space).
If I is homogeneous then $\{M + I \mid M \in \Delta(I), \deg(M) = d\}$ is a basis for $k[\mathbf{X}]_d/I(d)$.
We also have that $M \in \Delta(I)$ if and only if M is not a multiple of $\mathrm{lm}(g_i)$ for all $i = 1, \ldots, s$.

Facts from Gröbner bases theory

Let $\mathcal{M}$ be the set of monomials of $k[X_1, \ldots, X_n] =: k[\mathbf{X}]$ and endow $\mathcal{M}$ with a monomial order $\prec$. Given $f \in k[\mathbf{X}] \setminus \{0\}$ the leading monomial of f ($\mathrm{lm}(f)$) is the greatest monomial appearing in f. Let $I \subset k[\mathbf{X}]$ be an ideal, we say that $\{g_1, \ldots, g_s\} \subset I$ is a Gröbner basis for I (w.r.t. $\prec$) if the leading monomial of any nonzero polynomial in I is multiple of $\mathrm{lm}(g_i)$ for some $i = 1, \ldots, s$. One can prove that such a set is a basis for I in the usual sense $I = (g_1, \ldots, g_s)$ and that any ideal has a Gröbner basis.
A related and very important concept is the *footprint* of I (w.r.t. $\prec$) defined as

$$\Delta(I) = \{M \in \mathcal{M} \mid M \text{ is not the leading monomial of any polynomial in } I\}$$

Buchberger (1965) proved that $\{M + I \mid M \in \Delta(I)\}$ is a basis for $k[\mathbf{X}]/I$ (considered as a k-vector space).
If I is homogeneous then $\{M + I \mid M \in \Delta(I), \deg(M) = d\}$ is a basis for $k[\mathbf{X}]_d/I(d)$.
We also have that $M \in \Delta(I)$ if and only if M is not a multiple of $\mathrm{lm}(g_i)$ for all $i = 1, \ldots, s$.

Facts from Gröbner bases theory

Let $\mathcal{M}$ be the set of monomials of $k[X_1, \ldots, X_n] =: k[\mathbf{X}]$ and endow $\mathcal{M}$ with a monomial order $\prec$. Given $f \in k[\mathbf{X}] \setminus \{0\}$ the leading monomial of f ($\mathrm{lm}(f)$) is the greatest monomial appearing in f. Let $I \subset k[\mathbf{X}]$ be an ideal, we say that $\{g_1, \ldots, g_s\} \subset I$ is a Gröbner basis for I (w.r.t. $\prec$) if the leading monomial of any nonzero polynomial in I is multiple of $\mathrm{lm}(g_i)$ for some $i = 1, \ldots, s$. One can prove that such a set is a basis for I in the usual sense $I = (g_1, \ldots, g_s)$ and that any ideal has a Gröbner basis.

A related and very important concept is the *footprint* of I (w.r.t. $\prec$) defined as

$\Delta(I) = \{M \in \mathcal{M} \mid M \text{ is not the leading monomial of any polynomial in } I\}$

Buchberger (1965) proved that $\{M + I \mid M \in \Delta(I)\}$ is a basis for $k[\mathbf{X}]/I$ (considered as a k-vector space).

If I is homogeneous then $\{M + I \mid M \in \Delta(I), \deg(M) = d\}$ is a basis for $k[\mathbf{X}]_d / I(d)$.

We also have that $M \in \Delta(I)$ if and only if M is not a multiple of $\mathrm{lm}(g_i)$ for all $i = 1, \ldots, s$.

Facts from Gröbner bases theory

Let $\mathcal{M}$ be the set of monomials of $k[X_1, \ldots, X_n] =: k[\mathbf{X}]$ and endow $\mathcal{M}$ with a monomial order $\prec$. Given $f \in k[\mathbf{X}] \setminus \{0\}$ the leading monomial of f ($\text{lm}(f)$) is the greatest monomial appearing in f. Let $I \subset k[\mathbf{X}]$ be an ideal, we say that $\{g_1, \ldots, g_s\} \subset I$ is a Gröbner basis for I (w.r.t. $\prec$) if the leading monomial of any nonzero polynomial in I is multiple of $\text{lm}(g_i)$ for some $i = 1, \ldots, s$. One can prove that such a set is a basis for I in the usual sense $I = (g_1, \ldots, g_s)$ and that any ideal has a Gröbner basis.

A related and very important concept is the *footprint* of I (w.r.t. $\prec$) defined as

$\Delta(I) = \{M \in \mathcal{M} \mid M \text{ is not the leading monomial of any polynomial in } I\}$

Buchberger (1965) proved that $\{M + I \mid M \in \Delta(I)\}$ is a basis for $k[\mathbf{X}]/I$ (considered as a k-vector space).

If I is homogeneous then $\{M + I \mid M \in \Delta(I), \deg(M) = d\}$ is a basis for $k[\mathbf{X}]_d/I(d)$.

We also have that $M \in \Delta(I)$ if and only if M is not a multiple of $\text{lm}(g_i)$ for all $i = 1, \ldots, s$.

Facts from Gröbner bases theory

Let $\mathcal{M}$ be the set of monomials of $k[X_1, \ldots, X_n] =: k[\mathbf{X}]$ and endow $\mathcal{M}$ with a monomial order $\prec$. Given $f \in k[\mathbf{X}] \setminus \{0\}$ the leading monomial of f ($\mathrm{lm}(f)$) is the greatest monomial appearing in f. Let $I \subset k[\mathbf{X}]$ be an ideal, we say that $\{g_1, \ldots, g_s\} \subset I$ is a Gröbner basis for I (w.r.t. $\prec$) if the leading monomial of any nonzero polynomial in I is multiple of $\mathrm{lm}(g_i)$ for some $i = 1, \ldots, s$. One can prove that such a set is a basis for I in the usual sense $I = (g_1, \ldots, g_s)$ and that any ideal has a Gröbner basis.

A related and very important concept is the *footprint* of I (w.r.t. $\prec$) defined as

$\Delta(I) = \{M \in \mathcal{M} \mid M$ is not the leading monomial of any polynomial in $I\}$

Buchberger (1965) proved that $\{M + I \mid M \in \Delta(I)\}$ is a basis for $k[\mathbf{X}]/I$ (considered as a k-vector space).

If I is homogeneous then $\{M + I \mid M \in \Delta(I), \deg(M) = d\}$ is a basis for $k[\mathbf{X}]_d / I(d)$.

We also have that $M \in \Delta(I)$ if and only if M is not a multiple of $\mathrm{lm}(g_i)$ for all $i = 1, \ldots, s$.

Facts from Gröbner bases theory

Let $\mathcal{M}$ be the set of monomials of $k[X_1, \ldots, X_n] =: k[\mathbf{X}]$ and endow $\mathcal{M}$ with a monomial order $\prec$. Given $f \in k[\mathbf{X}] \setminus \{0\}$ the leading monomial of f ($\text{lm}(f)$) is the greatest monomial appearing in f. Let $I \subset k[\mathbf{X}]$ be an ideal, we say that $\{g_1, \ldots, g_s\} \subset I$ is a Gröbner basis for I (w.r.t. $\prec$) if the leading monomial of any nonzero polynomial in I is multiple of $\text{lm}(g_i)$ for some $i = 1, \ldots, s$. One can prove that such a set is a basis for I in the usual sense $I = (g_1, \ldots, g_s)$ and that any ideal has a Gröbner basis.

A related and very important concept is the *footprint* of I (w.r.t. $\prec$) defined as

$\Delta(I) = \{M \in \mathcal{M} \mid M \text{ is not the leading monomial of any polynomial in } I\}$

Buchberger (1965) proved that $\{M + I \mid M \in \Delta(I)\}$ is a basis for $k[\mathbf{X}]/I$ (considered as a k-vector space).

If I is homogeneous then $\{M + I \mid M \in \Delta(I), \deg(M) = d\}$ is a basis for $k[\mathbf{X}]_d/I(d)$.

We also have that $M \in \Delta(I)$ if and only if M is not a multiple of $\text{lm}(g_i)$ for all $i = 1, \ldots, s$.

Facts from Gröbner bases theory

Let $\mathcal{M}$ be the set of monomials of $k[X_1, \ldots, X_n] =: k[\mathbf{X}]$ and endow $\mathcal{M}$ with a monomial order $\prec$. Given $f \in k[\mathbf{X}] \setminus \{0\}$ the leading monomial of f ($\text{lm}(f)$) is the greatest monomial appearing in f. Let $I \subset k[\mathbf{X}]$ be an ideal, we say that $\{g_1, \ldots, g_s\} \subset I$ is a Gröbner basis for I (w.r.t. $\prec$) if the leading monomial of any nonzero polynomial in I is multiple of $\text{lm}(g_i)$ for some $i = 1, \ldots, s$. One can prove that such a set is a basis for I in the usual sense $I = (g_1, \ldots, g_s)$ and that any ideal has a Gröbner basis.
A related and very important concept is the *footprint* of I (w.r.t. $\prec$) defined as

$$\Delta(I) = \{M \in \mathcal{M} \mid M \text{ is not the leading monomial of any polynomial in } I\}$$

Buchberger (1965) proved that $\{M + I \mid M \in \Delta(I)\}$ is a basis for $k[\mathbf{X}]/I$ (considered as a k-vector space).
If I is homogeneous then $\{M + I \mid M \in \Delta(I), \deg(M) = d\}$ is a basis for $k[\mathbf{X}]_d / I(d)$.
We also have that $M \in \Delta(I)$ if and only if M is not a multiple of $\text{lm}(g_i)$ for all $i = 1, \ldots, s$.

Facts from Gröbner bases theory

Let $\mathcal{M}$ be the set of monomials of $k[X_1, \ldots, X_n] =: k[\mathbf{X}]$ and endow $\mathcal{M}$ with a monomial order $\prec$. Given $f \in k[\mathbf{X}] \setminus \{0\}$ the leading monomial of f ($\text{lm}(f)$) is the greatest monomial appearing in f. Let $I \subset k[\mathbf{X}]$ be an ideal, we say that $\{g_1, \ldots, g_s\} \subset I$ is a Gröbner basis for I (w.r.t. $\prec$) if the leading monomial of any nonzero polynomial in I is multiple of $\text{lm}(g_i)$ for some $i = 1, \ldots, s$. One can prove that such a set is a basis for I in the usual sense $I = (g_1, \ldots, g_s)$ and that any ideal has a Gröbner basis.
A related and very important concept is the *footprint* of I (w.r.t. $\prec$) defined as
$\Delta(I) = \{M \in \mathcal{M} \mid M \text{ is not the leading monomial of any polynomial in } I\}$
Buchberger (1965) proved that $\{M + I \mid M \in \Delta(I)\}$ is a basis for $k[\mathbf{X}]/I$ (considered as a k-vector space).
If I is homogeneous then $\{M + I \mid M \in \Delta(I), \deg(M) = d\}$ is a basis for $k[\mathbf{X}]_d/I(d)$.
We also have that $M \in \Delta(I)$ if and only if M is not a multiple of $\text{lm}(g_i)$ for all $i = 1, \ldots, s$.

Facts from Gröbner bases theory

Let $\mathcal{M}$ be the set of monomials of $k[X_1, \ldots, X_n] =: k[\mathbf{X}]$ and endow $\mathcal{M}$ with a monomial order $\prec$. Given $f \in k[\mathbf{X}] \setminus \{0\}$ the leading monomial of f ($\mathrm{lm}(f)$) is the greatest monomial appearing in f. Let $I \subset k[\mathbf{X}]$ be an ideal, we say that $\{g_1, \ldots, g_s\} \subset I$ is a Gröbner basis for I (w.r.t. $\prec$) if the leading monomial of any nonzero polynomial in I is multiple of $\mathrm{lm}(g_i)$ for some $i = 1, \ldots, s$. One can prove that such a set is a basis for I in the usual sense $I = (g_1, \ldots, g_s)$ and that any ideal has a Gröbner basis.
A related and very important concept is the *footprint* of I (w.r.t. $\prec$) defined as

$\Delta(I) = \{M \in \mathcal{M} \mid M$ is not the leading monomial of any polynomial in $I\}$

Buchberger (1965) proved that $\{M + I \mid M \in \Delta(I)\}$ is a basis for $k[\mathbf{X}]/I$ (considered as a k-vector space).
If I is homogeneous then $\{M + I \mid M \in \Delta(I), \deg(M) = d\}$ is a basis for $k[\mathbf{X}]_d/I(d)$.
We also have that $M \in \Delta(I)$ if and only if M is not a multiple of $\mathrm{lm}(g_i)$ for all $i = 1, \ldots, s$.

Facts from Gröbner bases theory

Let $\mathcal{M}$ be the set of monomials of $k[X_1, \ldots, X_n] =: k[\mathbf{X}]$ and endow $\mathcal{M}$ with a monomial order $\prec$. Given $f \in k[\mathbf{X}] \setminus \{0\}$ the leading monomial of f ($\text{lm}(f)$) is the greatest monomial appearing in f. Let $I \subset k[\mathbf{X}]$ be an ideal, we say that $\{g_1, \ldots, g_s\} \subset I$ is a Gröbner basis for I (w.r.t. $\prec$) if the leading monomial of any nonzero polynomial in I is multiple of $\text{lm}(g_i)$ for some $i = 1, \ldots, s$. One can prove that such a set is a basis for I in the usual sense $I = (g_1, \ldots, g_s)$ and that any ideal has a Gröbner basis.
A related and very important concept is the *footprint* of I (w.r.t. $\prec$) defined as

$\Delta(I) = \{M \in \mathcal{M} \mid M$ is not the leading monomial of any polynomial in $I\}$

Buchberger (1965) proved that $\{M + I \mid M \in \Delta(I)\}$ is a basis for $k[\mathbf{X}]/I$ (considered as a k-vector space).
If I is homogeneous then $\{M + I \mid M \in \Delta(I), \deg(M) = d\}$ is a basis for $k[\mathbf{X}]_d/I(d)$.
We also have that $M \in \Delta(I)$ if and only if M is not a multiple of $\text{lm}(g_i)$ for all $i = 1, \ldots, s$.

Facts from Gröbner bases theory

Let $\mathcal{M}$ be the set of monomials of $k[X_1, \ldots, X_n] =: k[\mathbf{X}]$ and endow $\mathcal{M}$ with a monomial order $\prec$. Given $f \in k[\mathbf{X}] \setminus \{0\}$ the leading monomial of f ($\mathrm{lm}(f)$) is the greatest monomial appearing in f. Let $I \subset k[\mathbf{X}]$ be an ideal, we say that $\{g_1, \ldots, g_s\} \subset I$ is a Gröbner basis for I (w.r.t. $\prec$) if the leading monomial of any nonzero polynomial in I is multiple of $\mathrm{lm}(g_i)$ for some $i = 1, \ldots, s$. One can prove that such a set is a basis for I in the usual sense $I = (g_1, \ldots, g_s)$ and that any ideal has a Gröbner basis.

A related and very important concept is the *footprint* of I (w.r.t. $\prec$) defined as

$\Delta(I) = \{M \in \mathcal{M} \mid M$ is not the leading monomial of any polynomial in $I\}$

Buchberger (1965) proved that $\{M + I \mid M \in \Delta(I)\}$ is a basis for $k[\mathbf{X}]/I$ (considered as a k-vector space).

If I is homogeneous then $\{M + I \mid$ ... r $k[\mathbf{X}]_d/I(d)$.

$$k[\mathbf{X}]/I = \bigoplus_{d=0}^{\infty} k[\mathbf{X}]_d/I(d)$$

We also have that $M \in \Delta(I)$ if and ... $)$ for all $i = 1, \ldots, s$.

Facts from Gröbner bases theory

Let $\mathcal{M}$ be the set of monomials of $k[X_1, \ldots, X_n] =: k[\mathbf{X}]$ and endow $\mathcal{M}$ with a monomial order $\prec$. Given $f \in k[\mathbf{X}] \setminus \{0\}$ the leading monomial of f ($\text{lm}(f)$) is the greatest monomial appearing in f. Let $I \subset k[\mathbf{X}]$ be an ideal, we say that $\{g_1, \ldots, g_s\} \subset I$ is a Gröbner basis for I (w.r.t. $\prec$) if the leading monomial of any nonzero polynomial in I is multiple of $\text{lm}(g_i)$ for some $i = 1, \ldots, s$. One can prove that such a set is a basis for I in the usual sense $I = (g_1, \ldots, g_s)$ and that any ideal has a Gröbner basis.
A related and very important concept is the *footprint* of I (w.r.t. $\prec$) defined as

$\Delta(I) = \{M \in \mathcal{M} \mid M \text{ is not the leading monomial of any polynomial in } I\}$

Buchberger (1965) proved that $\{M + I \mid M \in \Delta(I)\}$ is a basis for $k[\mathbf{X}]/I$ (considered as a k-vector space).
If I is homogeneous then $\{M + I \mid M \in \Delta(I), \deg(M) = d\}$ is a basis for $k[\mathbf{X}]_d / I(d)$.
We also have that $M \in \Delta(I)$ if and only if M is not a multiple of $\text{lm}(g_i)$ for all $i = 1, \ldots, s$.

Facts from Gröbner bases theory

Let $\mathcal{M}$ be the set of monomials of $k[X_1, \ldots, X_n] =: k[\mathbf{X}]$ and endow $\mathcal{M}$ with a monomial order $\prec$. Given $f \in k[\mathbf{X}] \setminus \{0\}$ the leading monomial of f ($\text{lm}(f)$) is the greatest monomial appearing in f. Let $I \subset k[\mathbf{X}]$ be an ideal, we say that $\{g_1, \ldots, g_s\} \subset I$ is a Gröbner basis for I (w.r.t. $\prec$) if the leading monomial of any nonzero polynomial in I is multiple of $\text{lm}(g_i)$ for some $i = 1, \ldots, s$. One can prove that such a set is a basis for I in the usual sense $I = (g_1, \ldots, g_s)$ and that any ideal has a Gröbner basis.
A related and very important concept is the *footprint* of I (w.r.t. $\prec$) defined as

$\Delta(I) = \{M \in \mathcal{M} \mid M \text{ is not the leading monomial of any polynomial in } I\}$

Buchberger (1965) proved that $\{M + I \mid M \in \Delta(I)\}$ is a basis for $k[\mathbf{X}]/I$ (considered as a k-vector space).
If I is homogeneous then $\{M + I \mid M \in \Delta(I), \deg(M) = d\}$ is a basis for $k[\mathbf{X}]_d / I(d)$.
We also have that $M \in \Delta(I)$ if and only if M is not a multiple of $\text{lm}(g_i)$ for all $i = 1, \ldots, s$.

Gröbner bases and coding theory

A (linear error correcting) code of length N defined over $\mathbb{F}_q$ is an $\mathbb{F}_q$-vector subspace $C \subset \mathbb{F}_q^N$. Given an N-tuple $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{F}_q^N$ the weight of α is $w(\alpha) = \#\{i \mid \alpha_i \neq 0\}$ and the minimum distance of C is $d_{\min}(C) := \min\{w(\alpha) \mid \alpha \in C \setminus \{\mathbf{0}\}\}$.

One may use a set $X = \{P_1, \ldots, P_N\} \subset \mathbb{A}^N(\mathbb{F}_q)$ to construct a code in the following way. Let $I_X \subset \mathbb{F}_q[X_0, \ldots, X_N] = \mathbb{F}_q[\mathbf{X}]$ be the ideal of X and let $\varphi : \mathbb{F}_q[\mathbf{X}]/I_X \to \mathbb{F}_q^N$ be given by $\varphi(f + I_X) = (f(P_1), \ldots, f(P_N))$. It is not difficult to show that φ is an isomorphism of $\mathbb{F}_q$-vector spaces. Thus, for any subspace $L \subset \mathbb{F}_q[\mathbf{X}]/I_X$ we have a code $C_L := \varphi(L)$.

Let d be a nonnegative integer and let

$L_d := \{f + I_X \mid f = 0 \text{ or } \deg(f) \leq d\}$.

In this case we say that C_{L_d} is "of Reed-Muller type" and has order d.

From Buchberger's result we know that the classes of the monomials in $\Delta(I_X))$ form a basis for $\mathbb{F}_q[\mathbf{X}]/I_X$ (in particular $\#(\Delta(I_X)) = N$) and one may prove that the set $\Delta(I_X)_d := \{M + I_X \mid M \in \Delta(I_X), \deg(M) \leq d\}$ is a basis for L_d, so that $\dim(C_{L_d}) = \#(\Delta(I_X)_d)$.

Gröbner bases and coding theory

A (linear error correcting) code of length N defined over $\mathbb{F}_q$ is an $\mathbb{F}_q$-vector subspace $C \subset \mathbb{F}_q^N$. Given an N-tuple $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_N) \in \mathbb{F}_q^N$ the weight of $\boldsymbol{\alpha}$ is $w(\boldsymbol{\alpha}) = \#\{i \mid \alpha_i \neq 0\}$ and the minimum distance of C is $d_{\min}(C) := \min\{w(\boldsymbol{\alpha}) \mid \boldsymbol{\alpha} \in C \setminus \{\mathbf{0}\}\}$.

One may use a set $X = \{P_1, \ldots, P_N\} \subset \mathbb{A}^N(\mathbb{F}_q)$ to construct a code in the following way. Let $I_X \subset \mathbb{F}_q[X_0, \ldots, X_N] = \mathbb{F}_q[\mathbf{X}]$ be the ideal of X and let $\varphi : \mathbb{F}_q[\mathbf{X}]/I_X \to \mathbb{F}_q^N$ be given by $\varphi(f + I_X) = (f(P_1), \ldots, f(P_N))$. It is not difficult to show that φ is an isomorphism of $\mathbb{F}_q$-vector spaces. Thus, for any subspace $L \subset \mathbb{F}_q[\mathbf{X}]/I_X$ we have a code $C_L := \varphi(L)$.

Let d be a nonnegative integer and let

$L_d := \{f + I_X \mid f = 0 \text{ or } \deg(f) \leq d\}$.

In this case we say that C_{L_d} is "of Reed-Muller type" and has order d.

From Buchberger's result we know that the classes of the monomials in $\Delta(I_X))$ form a basis for $\mathbb{F}_q[\mathbf{X}]/I_X$ (in particular $\#(\Delta(I_X)) = N$) and one may prove that the set $\Delta(I_X)_d := \{M + I_X \mid M \in \Delta(I_X), \deg(M) \leq d\}$ is a basis for L_d, so that $\dim(C_{L_d}) = \#(\Delta(I_X)_d)$.

Gröbner bases and coding theory

A (linear error correcting) code of length N defined over $\mathbb{F}_q$ is an $\mathbb{F}_q$-vector subspace $C \subset \mathbb{F}_q^N$. Given an N-tuple $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_N) \in \mathbb{F}_q^N$ the weight of $\boldsymbol{\alpha}$ is $w(\boldsymbol{\alpha}) = \#\{i \mid \alpha_i \neq 0\}$ and the minimum distance of C is $d_{\min}(C) := \min\{w(\boldsymbol{\alpha}) \mid \boldsymbol{\alpha} \in C \setminus \{\mathbf{0}\}\}$.

One may use a set $X = \{P_1, \ldots, P_N\} \subset \mathbb{A}^N(\mathbb{F}_q)$ to construct a code in the following way. Let $I_X \subset \mathbb{F}_q[X_0, \ldots, X_N] = \mathbb{F}_q[\mathbf{X}]$ be the ideal of X and let $\varphi : \mathbb{F}_q[\mathbf{X}]/I_X \to \mathbb{F}_q^N$ be given by $\varphi(f + I_X) = (f(P_1), \ldots, f(P_N))$. It is not difficult to show that φ is an isomorphism of $\mathbb{F}_q$-vector spaces. Thus, for any subspace $L \subset \mathbb{F}_q[\mathbf{X}]/I_X$ we have a code $C_L := \varphi(L)$.

Let d be a nonnegative integer and let
$L_d := \{f + I_X \mid f = 0 \text{ or } \deg(f) \leq d\}$.

In this case we say that C_{L_d} is "of Reed-Muller type" and has order d.

From Buchberger's result we know that the classes of the monomials in $\Delta(I_X))$ form a basis for $\mathbb{F}_q[\mathbf{X}]/I_X$ (in particular $\#(\Delta(I_X)) = N$) and one may prove that the set $\Delta(I_X)_d := \{M + I_X \mid M \in \Delta(I_X), \deg(M) \leq d\}$ is a basis for L_d, so that $\dim(C_{L_d}) = \#(\Delta(I_X)_d)$.

Gröbner bases and coding theory

A (linear error correcting) code of length N defined over $\mathbb{F}_q$ is an $\mathbb{F}_q$-vector subspace $C \subset \mathbb{F}_q^N$. Given an N-tuple $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_N) \in \mathbb{F}_q^N$ the weight of $\boldsymbol{\alpha}$ is $w(\boldsymbol{\alpha}) = \#\{i \,|\, \alpha_i \neq 0\}$ and the minimum distance of C is $d_{\min}(C) := \min\{w(\boldsymbol{\alpha}) \,|\, \boldsymbol{\alpha} \in C \setminus \{\mathbf{0}\}\}$.
One may use a set $X = \{P_1, \ldots, P_N\} \subset \mathbb{A}^N(\mathbb{F}_q)$ to construct a code in the following way. Let $I_X \subset \mathbb{F}_q[X_0, \ldots, X_N] = \mathbb{F}_q[\mathbf{X}]$ be the ideal of X and let $\varphi : \mathbb{F}_q[\mathbf{X}]/I_X \to \mathbb{F}_q^N$ be given by $\varphi(f + I_X) = (f(P_1), \ldots, f(P_N))$.
It is not difficult to show that φ is an isomorphism of $\mathbb{F}_q$-vector spaces.
Thus, for any subspace $L \subset \mathbb{F}_q[\mathbf{X}]/I_X$ we have a code $C_L := \varphi(L)$.
Let d be a nonnegative integer and let
$L_d := \{f + I_X \,|\, f = 0 \text{ or } \deg(f) \leq d\}$.
In this case we say that C_{L_d} is "of Reed-Muller type" and has order d.
From Buchberger's result we know that the classes of the monomials in $\Delta(I_X))$ form a basis for $\mathbb{F}_q[\mathbf{X}]/I_X$ (in particular $\#(\Delta(I_X)) = N$) and one may prove that the set $\Delta(I_X)_d := \{M + I_X \,|\, M \in \Delta(I_X), \deg(M) \leq d\}$ is a basis for L_d, so that $\dim(C_{L_d}) = \#(\Delta(I_X)_d)$.

Gröbner bases and coding theory

A (linear error correcting) code of length N defined over $\mathbb{F}_q$ is an $\mathbb{F}_q$-vector subspace $C \subset \mathbb{F}_q^N$. Given an N-tuple $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_N) \in \mathbb{F}_q^N$ the weight of $\boldsymbol{\alpha}$ is $w(\boldsymbol{\alpha}) = \#\{i \mid \alpha_i \neq 0\}$ and the minimum distance of C is $d_{\min}(C) := \min\{w(\boldsymbol{\alpha}) \mid \boldsymbol{\alpha} \in C \setminus \{\mathbf{0}\}\}$.

One may use a set $X = \{P_1, \ldots, P_N\} \subset \mathbb{A}^N(\mathbb{F}_q)$ to construct a code in the following way. Let $I_X \subset \mathbb{F}_q[X_0, \ldots, X_N] = \mathbb{F}_q[\mathbf{X}]$ be the ideal of X and let $\varphi : \mathbb{F}_q[\mathbf{X}]/I_X \to \mathbb{F}_q^N$ be given by $\varphi(f + I_X) = (f(P_1), \ldots, f(P_N))$.

It is not difficult to show that φ is an isomorphism of $\mathbb{F}_q$-vector spaces. Thus, for any subspace $L \subset \mathbb{F}_q[\mathbf{X}]/I_X$ we have a code $C_L := \varphi(L)$.

Let d be a nonnegative integer and let

$L_d := \{f + I_X \mid f = 0 \text{ or } \deg(f) \leq d\}$.

In this case we say that C_{L_d} is "of Reed-Muller type" and has order d.

From Buchberger's result we know that the classes of the monomials in $\Delta(I_X))$ form a basis for $\mathbb{F}_q[\mathbf{X}]/I_X$ (in particular $\#(\Delta(I_X)) = N$) and one may prove that the set $\Delta(I_X)_d := \{M + I_X \mid M \in \Delta(I_X), \deg(M) \leq d\}$ is a basis for L_d, so that $\dim(C_{L_d}) = \#(\Delta(I_X)_d)$.

Gröbner bases and coding theory

A (linear error correcting) code of length N defined over $\mathbb{F}_q$ is an $\mathbb{F}_q$-vector subspace $C \subset \mathbb{F}_q^N$. Given an N-tuple $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_N) \in \mathbb{F}_q^N$ the weight of $\boldsymbol{\alpha}$ is $w(\boldsymbol{\alpha}) = \#\{i \mid \alpha_i \neq 0\}$ and the minimum distance of C is $d_{\min}(C) := \min\{w(\boldsymbol{\alpha}) \mid \boldsymbol{\alpha} \in C \setminus \{\mathbf{0}\}\}$.

One may use a set $X = \{P_1, \ldots, P_N\} \subset \mathbb{A}^N(\mathbb{F}_q)$ to construct a code in the following way. Let $I_X \subset \mathbb{F}_q[X_0, \ldots, X_N] = \mathbb{F}_q[\mathbf{X}]$ be the ideal of X and let $\varphi : \mathbb{F}_q[\mathbf{X}]/I_X \to \mathbb{F}_q^N$ be given by $\varphi(f + I_X) = (f(P_1), \ldots, f(P_N))$.

It is not difficult to show that φ is an isomorphism of $\mathbb{F}_q$-vector spaces. Thus, for any subspace $L \subset \mathbb{F}_q[\mathbf{X}]/I_X$ we have a code $C_L := \varphi(L)$.

Let d be a nonnegative integer and let
$L_d := \{f + I_X \mid f = 0 \text{ or } \deg(f) \leq d\}$.

In this case we say that C_{L_d} is "of Reed-Muller type" and has order d.

From Buchberger's result we know that the classes of the monomials in $\Delta(I_X))$ form a basis for $\mathbb{F}_q[\mathbf{X}]/I_X$ (in particular $\#(\Delta(I_X)) = N$) and one may prove that the set $\Delta(I_X)_d := \{M + I_X \mid M \in \Delta(I_X), \deg(M) \leq d\}$ is a basis for L_d, so that $\dim(C_{L_d}) = \#(\Delta(I_X)_d)$.

Gröbner bases and coding theory

A (linear error correcting) code of length N defined over $\mathbb{F}_q$ is an $\mathbb{F}_q$-vector subspace $C \subset \mathbb{F}_q^N$. Given an N-tuple $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_N) \in \mathbb{F}_q^N$ the weight of $\boldsymbol{\alpha}$ is $w(\boldsymbol{\alpha}) = \#\{i \mid \alpha_i \neq 0\}$ and the minimum distance of C is $d_{\min}(C) := \min\{w(\boldsymbol{\alpha}) \mid \boldsymbol{\alpha} \in C \setminus \{\mathbf{0}\}\}$.
One may use a set $X = \{P_1, \ldots, P_N\} \subset \mathbb{A}^N(\mathbb{F}_q)$ to construct a code in the following way. Let $I_X \subset \mathbb{F}_q[X_0, \ldots, X_N] = \mathbb{F}_q[\mathbf{X}]$ be the ideal of X and let $\varphi : \mathbb{F}_q[\mathbf{X}]/I_X \to \mathbb{F}_q^N$ be given by $\varphi(f + I_X) = (f(P_1), \ldots, f(P_N))$.
It is not difficult to show that φ is an isomorphism of $\mathbb{F}_q$-vector spaces.
Thus, for any subspace $L \subset \mathbb{F}_q[\mathbf{X}]/I_X$ we have a code $C_L := \varphi(L)$.
Let d be a nonnegative integer and let
$L_d := \{f + I_X \mid f = 0 \text{ or } \deg(f) \leq d\}$.
In this case we say that C_{L_d} is "of Reed-Muller type" and has order d.
From Buchberger's result we know that the classes of the monomials in $\Delta(I_X))$ form a basis for $\mathbb{F}_q[\mathbf{X}]/I_X$ (in particular $\#(\Delta(I_X)) = N$) and one may prove that the set $\Delta(I_X)_d := \{M + I_X \mid M \in \Delta(I_X), \deg(M) \leq d\}$ is a basis for L_d, so that $\dim(C_{L_d}) = \#(\Delta(I_X)_d)$.

Gröbner bases and coding theory

A (linear error correcting) code of length N defined over $\mathbb{F}_q$ is an $\mathbb{F}_q$-vector subspace $C \subset \mathbb{F}_q^N$. Given an N-tuple $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_N) \in \mathbb{F}_q^N$ the weight of $\boldsymbol{\alpha}$ is $w(\boldsymbol{\alpha}) = \#\{i \,|\, \alpha_i \neq 0\}$ and the minimum distance of C is $d_{\min}(C) := \min\{w(\boldsymbol{\alpha}) \,|\, \boldsymbol{\alpha} \in C \setminus \{\mathbf{0}\}\}$.
One may use a set $X = \{P_1, \ldots, P_N\} \subset \mathbb{A}^N(\mathbb{F}_q)$ to construct a code in the following way. Let $I_X \subset \mathbb{F}_q[X_0, \ldots, X_N] = \mathbb{F}_q[\mathbf{X}]$ be the ideal of X and let $\varphi : \mathbb{F}_q[\mathbf{X}]/I_X \to \mathbb{F}_q^N$ be given by $\varphi(f + I_X) = (f(P_1), \ldots, f(P_N))$.
It is not difficult to show that φ is an isomorphism of $\mathbb{F}_q$-vector spaces.
Thus, for any subspace $L \subset \mathbb{F}_q[\mathbf{X}]/I_X$ we have a code $C_L := \varphi(L)$.
Let d be a nonnegative integer and let
$L_d := \{f + I_X \,|\, f = 0 \text{ or } \deg(f) \leq d\}$.
In this case we say that C_{L_d} is "of Reed-Muller type" and has order d.
From Buchberger's result we know that the classes of the monomials in $\Delta(I_X))$ form a basis for $\mathbb{F}_q[\mathbf{X}]/I_X$ (in particular $\#(\Delta(I_X)) = N$) and one may prove that the set $\Delta(I_X)_d := \{M + I_X \,|\, M \in \Delta(I_X), \deg(M) \leq d\}$ is a basis for L_d, so that $\dim(C_{L_d}) = \#(\Delta(I_X)_d)$.

Gröbner bases and coding theory

A (linear error correcting) code of length N defined over $\mathbb{F}_q$ is an $\mathbb{F}_q$-vector subspace $C \subset \mathbb{F}_q^N$. Given an N-tuple $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_N) \in \mathbb{F}_q^N$ the weight of $\boldsymbol{\alpha}$ is $w(\boldsymbol{\alpha}) = \#\{i \,|\, \alpha_i \neq 0\}$ and the minimum distance of C is $d_{\min}(C) := \min\{w(\boldsymbol{\alpha}) \,|\, \boldsymbol{\alpha} \in C \setminus \{\mathbf{0}\}\}$.
One may use a set $X = \{P_1, \ldots, P_N\} \subset \mathbb{A}^N(\mathbb{F}_q)$ to construct a code in the following way. Let $I_X \subset \mathbb{F}_q[X_0, \ldots, X_N] = \mathbb{F}_q[\mathbf{X}]$ be the ideal of X and let $\varphi : \mathbb{F}_q[\mathbf{X}]/I_X \to \mathbb{F}_q^N$ be given by $\varphi(f + I_X) = (f(P_1), \ldots, f(P_N))$.
It is not difficult to show that φ is an isomorphism of $\mathbb{F}_q$-vector spaces.
Thus, for any subspace $L \subset \mathbb{F}_q[\mathbf{X}]/I_X$ we have a code $C_L := \varphi(L)$.
Let d be a nonnegative integer and let
$L_d := \{f + I_X \,|\, f = 0 \text{ or } \deg(f) \leq d\}$.
In this case we say that C_{L_d} is "of Reed-Muller type" and has order d.
From Buchberger's result we know that the classes of the monomials in $\Delta(I_X))$ form a basis for $\mathbb{F}_q[\mathbf{X}]/I_X$ (in particular $\#(\Delta(I_X)) = N$) and one may prove that the set $\Delta(I_X)_d := \{M + I_X \,|\, M \in \Delta(I_X), \deg(M) \leq d\}$ is a basis for L_d, so that $\dim(C_{L_d}) = \#(\Delta(I_X)_d)$.

Gröbner bases and coding theory

A (linear error correcting) code of length N defined over $\mathbb{F}_q$ is an $\mathbb{F}_q$-vector subspace $C \subset \mathbb{F}_q^N$. Given an N-tuple $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_N) \in \mathbb{F}_q^N$ the weight of $\boldsymbol{\alpha}$ is $w(\boldsymbol{\alpha}) = \#\{i \mid \alpha_i \neq 0\}$ and the minimum distance of C is $d_{\min}(C) := \min\{w(\boldsymbol{\alpha}) \mid \boldsymbol{\alpha} \in C \setminus \{\mathbf{0}\}\}$.

One may use a set $X = \{P_1, \ldots, P_N\} \subset \mathbb{A}^N(\mathbb{F}_q)$ to construct a code in the following way. Let $I_X \subset \mathbb{F}_q[X_0, \ldots, X_N] = \mathbb{F}_q[\mathbf{X}]$ be the ideal of X and let $\varphi : \mathbb{F}_q[\mathbf{X}]/I_X \to \mathbb{F}_q^N$ be given by $\varphi(f + I_X) = (f(P_1), \ldots, f(P_N))$. It is not difficult to show that φ is an isomorphism of $\mathbb{F}_q$-vector spaces. Thus, for any subspace $L \subset \mathbb{F}_q[\mathbf{X}]/I_X$ we have a code $C_L := \varphi(L)$.

Let d be a nonnegative integer and let

$L_d := \{f + I_X \mid f = 0 \text{ or } \deg(f) \leq d\}$.

In this case we say that C_{L_d} is "of Reed-Muller type" and has order d.

From Buchberger's result we know that the classes of the monomials in $\Delta(I_X))$ form a basis for $\mathbb{F}_q[\mathbf{X}]/I_X$ (in particular $\#(\Delta(I_X)) = N$) and one may prove that the set $\Delta(I_X)_d := \{M + I_X \mid M \in \Delta(I_X), \deg(M) \leq d\}$ is a basis for L_d, so that $\dim(C_{L_d}) = \#(\Delta(I_X)_d)$.

Gröbner bases and coding theory

A (linear error correcting) code of length N defined over $\mathbb{F}_q$ is an $\mathbb{F}_q$-vector subspace $C \subset \mathbb{F}_q^N$. Given an N-tuple $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_N) \in \mathbb{F}_q^N$ the weight of $\boldsymbol{\alpha}$ is $w(\boldsymbol{\alpha}) = \#\{i \mid \alpha_i \neq 0\}$ and the minimum distance of C is $d_{\min}(C) := \min\{w(\boldsymbol{\alpha}) \mid \boldsymbol{\alpha} \in C \setminus \{\mathbf{0}\}\}$.

One may use a set $X = \{P_1, \ldots, P_N\} \subset \mathbb{A}^N(\mathbb{F}_q)$ to construct a code in the following way. Let $I_X \subset \mathbb{F}_q[X_0, \ldots, X_N] = \mathbb{F}_q[\mathbf{X}]$ be the ideal of X and let $\varphi : \mathbb{F}_q[\mathbf{X}]/I_X \to \mathbb{F}_q^N$ be given by $\varphi(f + I_X) = (f(P_1), \ldots, f(P_N))$. It is not difficult to show that φ is an isomorphism of $\mathbb{F}_q$-vector spaces. Thus, for any subspace $L \subset \mathbb{F}_q[\mathbf{X}]/I_X$ we have a code $C_L := \varphi(L)$.

Let d be a nonnegative integer and let

$L_d := \{f + I_X \mid f = 0 \text{ or } \deg(f) \leq d\}$.

In this case we say that C_{L_d} is "of Reed-Muller type" and has order d.

From Buchberger's result we know that the classes of the monomials in $\Delta(I_X))$ form a basis for $\mathbb{F}_q[\mathbf{X}]/I_X$ (in particular $\#(\Delta(I_X)) = N$) and one may prove that the set $\Delta(I_X)_d := \{M + I_X \mid M \in \Delta(I_X), \deg(M) \leq d\}$ is a basis for L_d, so that $\dim(C_{L_d}) = \#(\Delta(I_X)_d)$.

Gröbner bases and coding theory

A (linear error correcting) code of length N defined over $\mathbb{F}_q$ is an $\mathbb{F}_q$-vector subspace $C \subset \mathbb{F}_q^N$. Given an N-tuple $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_N) \in \mathbb{F}_q^N$ the weight of $\boldsymbol{\alpha}$ is $w(\boldsymbol{\alpha}) = \#\{i \mid \alpha_i \neq 0\}$ and the minimum distance of C is $d_{\min}(C) := \min\{w(\boldsymbol{\alpha}) \mid \boldsymbol{\alpha} \in C \setminus \{\mathbf{0}\}\}$.

One may use a set $X = \{P_1, \ldots, P_N\} \subset \mathbb{A}^N(\mathbb{F}_q)$ to construct a code in the following way. Let $I_X \subset \mathbb{F}_q[X_0, \ldots, X_N] = \mathbb{F}_q[\mathbf{X}]$ be the ideal of X and let $\varphi : \mathbb{F}_q[\mathbf{X}]/I_X \to \mathbb{F}_q^N$ be given by $\varphi(f + I_X) = (f(P_1), \ldots, f(P_N))$.

It is not difficult to show that φ is an isomorphism of $\mathbb{F}_q$-vector spaces. Thus, for any subspace $L \subset \mathbb{F}_q[\mathbf{X}]/I_X$ we have a code $C_L := \varphi(L)$.

Let d be a nonnegative integer and let

$L_d := \{f + I_X \mid f = 0 \text{ or } \deg(f) \leq d\}$.

In this case we say that C_{L_d} is "of Reed-Muller type" and has order d.

From Buchberger's result we know that the classes of the monomials in $\Delta(I_X))$ form a basis for $\mathbb{F}_q[\mathbf{X}]/I_X$ (in particular $\#(\Delta(I_X)) = N$) and one may prove that the set $\Delta(I_X)_d := \{M + I_X \mid M \in \Delta(I_X), \deg(M) \leq d\}$ is a basis for L_d, so that $\dim(C_{L_d}) = \#(\Delta(I_X)_d)$.

Gröbner bases and coding theory

A (linear error correcting) code of length N defined over $\mathbb{F}_q$ is an $\mathbb{F}_q$-vector subspace $C \subset \mathbb{F}_q^N$. Given an N-tuple $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_N) \in \mathbb{F}_q^N$ the weight of $\boldsymbol{\alpha}$ is $w(\boldsymbol{\alpha}) = \#\{i \mid \alpha_i \neq 0\}$ and the minimum distance of C is $d_{\min}(C) := \min\{w(\boldsymbol{\alpha}) \mid \boldsymbol{\alpha} \in C \setminus \{\mathbf{0}\}\}$.
One may use a set $X = \{P_1, \ldots, P_N\} \subset \mathbb{A}^N(\mathbb{F}_q)$ to construct a code in the following way. Let $I_X \subset \mathbb{F}_q[X_0, \ldots, X_N] = \mathbb{F}_q[\mathbf{X}]$ be the ideal of X and let $\varphi : \mathbb{F}_q[\mathbf{X}]/I_X \to \mathbb{F}_q^N$ be given by $\varphi(f + I_X) = (f(P_1), \ldots, f(P_N))$.
It is not difficult to show that φ is an isomorphism of $\mathbb{F}_q$-vector spaces.
Thus, for any subspace $L \subset \mathbb{F}_q[\mathbf{X}]/I_X$ we have a code $C_L := \varphi(L)$.
Let d be a nonnegative integer and let
$L_d := \{f + I_X \mid f = 0 \text{ or } \deg(f) \leq d\}$.
In this case we say that C_{L_d} is “of Reed-Muller type” and has order d.
From Buchberger's result we know that the classes of the monomials in $\Delta(I_X))$ form a basis for $\mathbb{F}_q[\mathbf{X}]/I_X$ (in particular $\#(\Delta(I_X)) = N$) and one may prove that the set $\Delta(I_X)_d := \{M + I_X \mid M \in \Delta(I_X), \deg(M) \leq d\}$ is a basis for L_d, so that $\dim(C_{L_d}) = \#(\Delta(I_X)_d)$.

Gröbner bases and coding theory

A (linear error correcting) code of length N defined over $\mathbb{F}_q$ is an $\mathbb{F}_q$-vector subspace $C \subset \mathbb{F}_q^N$. Given an N-tuple $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_N) \in \mathbb{F}_q^N$ the weight of $\boldsymbol{\alpha}$ is $w(\boldsymbol{\alpha}) = \#\{i \mid \alpha_i \neq 0\}$ and the minimum distance of C is $d_{\min}(C) := \min\{w(\boldsymbol{\alpha}) \mid \boldsymbol{\alpha} \in C \setminus \{\mathbf{0}\}\}$.
One may use a set $X = \{P_1, \ldots, P_N\} \subset \mathbb{A}^N(\mathbb{F}_q)$ to construct a code in the following way. Let $I_X \subset \mathbb{F}_q[X_0, \ldots, X_N] = \mathbb{F}_q[\mathbf{X}]$ be the ideal of X and let $\varphi : \mathbb{F}_q[\mathbf{X}]/I_X \to \mathbb{F}_q^N$ be given by $\varphi(f + I_X) = (f(P_1), \ldots, f(P_N))$.
It is not difficult to show that φ is an isomorphism of $\mathbb{F}_q$-vector spaces.
Thus, for any subspace $L \subset \mathbb{F}_q[\mathbf{X}]/I_X$ we have a code $C_L := \varphi(L)$.
Let d be a nonnegative integer and let
$L_d := \{f + I_X \mid f = 0 \text{ or } \deg(f) \leq d\}$.
In this case we say that C_{L_d} is "of Reed-Muller type" and has order d.
From Buchberger's result we know that the classes of the monomials in $\Delta(I_X))$ form a basis for $\mathbb{F}_q[\mathbf{X}]/I_X$ (in particular $\#(\Delta(I_X)) = N$) and one may prove that the set $\Delta(I_X)_d := \{M + I_X \mid M \in \Delta(I_X), \deg(M) \leq d\}$ is a basis for L_d, so that $\dim(C_{L_d}) = \#(\Delta(I_X)_d)$.

Gröbner basis methods and the parameters of C_L

As for the minimum distance $d_{\min}(C_L)$, we would like to estimate the number of zero entries in $\varphi(f + I_X) = (f(P_1), \ldots, f(P_N))$. Let $I_{X,f} := I_X + (f)$, we want to estimate $N - \#(V(I_{X,f}))$. From $I_X \subset I_{X,f}$ we get $\Delta(I_{X,f}) \subset \Delta(I_X)$, in particular $\Delta(I_{X,f})$ is finite which implies $\#(V(I_{X,f})) \leq \#(\Delta(I_{X,f}))$ and we get $N - \#(V(I_{X,f})) \geq N - \#(\Delta(I_{X,f}))$.

From Buchberger's result we can assume that f is a linear combination of monomials in $\Delta(I_X)$ so that $\mathrm{lm}(f) \in \Delta(I_X)$. It is not difficult to prove that $N - \#(\Delta(I_{X,f})) \geq \{M \in \Delta(I_X) \mid \mathrm{lm}(f) | M\}$, so the idea now is to determine for each monomial $M' \in \Delta(I_X)$ the cardinality of the set $\{M \in \Delta(I_X) \mid M' | M\}$, and from this determine a lower bound for $d_{\min}(C_L)$. Moreover, it is true that if $\{g_1, \ldots, g_s\}$ is a Gröbner basis for I_X and $\{f, g_1, \ldots, g_s\}$ is a Gröbner basis for $I_{X,f}$ then this bound (with $M' = \mathrm{lm}(f)$) is the true value of the minimum distance.

These techniques have been used to determine the parameters of codes C_{L_d} when $X = A_1 \times \cdots \times A_N \in \mathbb{A}^N(\mathbb{F}_q)$, as well as some higher Hamming weights for these codes.

Gröbner basis methods and the parameters of C_L

As for the minimum distance $d_{\min}(C_L)$, we would like to estimate the number of zero entries in $\varphi(f + I_X) = (f(P_1), \ldots, f(P_N))$. Let $I_{X,f} := I_X + (f)$, we want to estimate $N - \#(V(I_{X,f}))$. From $I_X \subset I_{X,f}$ we get $\Delta(I_{X,f}) \subset \Delta(I_X)$, in particular $\Delta(I_{X,f})$ is finite which implies $\#(V(I_{X,f})) \leq \#(\Delta(I_{X,f}))$ and we get $N - \#(V(I_{X,f})) \geq N - \#(\Delta(I_{X,f}))$.

From Buchberger's result we can assume that f is a linear combination of monomials in $\Delta(I_X)$ so that $\text{lm}(f) \in \Delta(I_X)$. It is not difficult to prove that $N - \#(\Delta(I_{X,f})) \geq \{M \in \Delta(I_X) \mid \text{lm}(f) | M\}$, so the idea now is to determine for each monomial $M' \in \Delta(I_X)$ the cardinality of the set $\{M \in \Delta(I_X) \mid M' | M\}$, and from this determine a lower bound for $d_{\min}(C_L)$. Moreover, it is true that if $\{g_1, \ldots, g_s\}$ is a Gröbner basis for I_X and $\{f, g_1, \ldots, g_s\}$ is a Gröbner basis for $I_{X,f}$ then this bound (with $M' = \text{lm}(f)$) is the true value of the minimum distance.

These techniques have been used to determine the parameters of codes C_{L_d} when $X = A_1 \times \cdots \times A_N \in \mathbb{A}^N(\mathbb{F}_q)$, as well as some higher Hamming weights for these codes.

Gröbner basis methods and the parameters of C_L

As for the minimum distance $d_{\min}(C_L)$, we would like to estimate the number of zero entries in $\varphi(f + I_X) = (f(P_1), \ldots, f(P_N))$. Let $I_{X,f} := I_X + (f)$, we want to estimate $N - \#(V(I_{X,f}))$. From $I_X \subset I_{X,f}$ we get $\Delta(I_{X,f}) \subset \Delta(I_X)$, in particular $\Delta(I_{X,f})$ is finite which implies $\#(V(I_{X,f})) \leq \#(\Delta(I_{X,f}))$ and we get $N - \#(V(I_{X,f})) \geq N - \#(\Delta(I_{X,f}))$.

From Buchberger's result we can assume that f is a linear combination of monomials in $\Delta(I_X)$ so that $\mathrm{lm}(f) \in \Delta(I_X)$. It is not difficult to prove that $N - \#(\Delta(I_{X,f})) \geq \{M \in \Delta(I_X) \mid \mathrm{lm}(f) | M\}$, so the idea now is to determine for each monomial $M' \in \Delta(I_X)$ the cardinality of the set $\{M \in \Delta(I_X) \mid M' | M\}$, and from this determine a lower bound for $d_{\min}(C_L)$. Moreover, it is true that if $\{g_1, \ldots, g_s\}$ is a Gröbner basis for I_X and $\{f, g_1, \ldots, g_s\}$ is a Gröbner basis for $I_{X,f}$ then this bound (with $M' = \mathrm{lm}(f)$) is the true value of the minimum distance.

These techniques have been used to determine the parameters of codes C_{L_d} when $X = A_1 \times \cdots \times A_N \in \mathbb{A}^N(\mathbb{F}_q)$, as well as some higher Hamming weights for these codes.

Gröbner basis methods and the parameters of C_L

As for the minimum distance $d_{\min}(C_L)$, we would like to estimate the number of zero entries in $\varphi(f + I_X) = (f(P_1), \ldots, f(P_N))$. Let $I_{X,f} := I_X + (f)$, we want to estimate $N - \#(V(I_{X,f}))$. From $I_X \subset I_{X,f}$ we get $\Delta(I_{X,f}) \subset \Delta(I_X)$, in particular $\Delta(I_{X,f})$ is finite which implies $\#(V(I_{X,f})) \leq \#(\Delta(I_{X,f}))$ and we get $N - \#(V(I_{X,f})) \geq N - \#(\Delta(I_{X,f}))$.

From Buchberger's result we can assume that f is a linear combination of monomials in $\Delta(I_X)$ so that $\mathrm{lm}(f) \in \Delta(I_X)$. It is not difficult to prove that $N - \#(\Delta(I_{X,f})) \geq \{M \in \Delta(I_X) \mid \mathrm{lm}(f) | M\}$, so the idea now is to determine for each monomial $M' \in \Delta(I_X)$ the cardinality of the set $\{M \in \Delta(I_X) \mid M' | M\}$, and from this determine a lower bound for $d_{\min}(C_L)$. Moreover, it is true that if $\{g_1, \ldots, g_s\}$ is a Gröbner basis for I_X and $\{f, g_1, \ldots, g_s\}$ is a Gröbner basis for $I_{X,f}$ then this bound (with $M' = \mathrm{lm}(f)$) is the true value of the minimum distance.

These techniques have been used to determine the parameters of codes C_{L_d} when $X = A_1 \times \cdots \times A_N \in \mathbb{A}^N(\mathbb{F}_q)$, as well as some higher Hamming weights for these codes.

Gröbner basis methods and the parameters of C_L

As for the minimum distance $d_{\min}(C_L)$, we would like to estimate the number of zero entries in $\varphi(f + I_X) = (f(P_1), \ldots, f(P_N))$. Let $I_{X,f} := I_X + (f)$, we want to estimate $N - \#(V(I_{X,f}))$. From $I_X \subset I_{X,f}$ we get $\Delta(I_{X,f}) \subset \Delta(I_X)$, in particular $\Delta(I_{X,f})$ is finite which implies $\#(V(I_{X,f})) \leq \#(\Delta(I_{X,f}))$ and we get $N - \#(V(I_{X,f})) \geq N - \#(\Delta(I_{X,f}))$. From Buchberger's result we can assume that f is a linear combination of monomials in $\Delta(I_X)$ so that $\text{lm}(f) \in \Delta(I_X)$. It is not difficult to prove that $N - \#(\Delta(I_{X,f})) \geq \{M \in \Delta(I_X) \mid \text{lm}(f) | M\}$, so the idea now is to determine for each monomial $M' \in \Delta(I_X)$ the cardinality of the set $\{M \in \Delta(I_X) \mid M' | M\}$, and from this determine a lower bound for $d_{\min}(C_L)$. Moreover, it is true that if $\{g_1, \ldots, g_s\}$ is a Gröbner basis for I_X and $\{f, g_1, \ldots, g_s\}$ is a Gröbner basis for $I_{X,f}$ then this bound (with $M' = \text{lm}(f)$) is the true value of the minimum distance.

These techniques have been used to determine the parameters of codes C_{L_d} when $X = A_1 \times \cdots \times A_N \in \mathbb{A}^N(\mathbb{F}_q)$, as well as some higher Hamming weights for these codes.

Gröbner basis methods and the parameters of C_L

As for the minimum distance $d_{\min}(C_L)$, we would like to estimate the number of zero entries in $\varphi(f + I_X) = (f(P_1), \ldots, f(P_N))$. Let $I_{X,f} := I_X + (f)$, we want to estimate $N - \#(V(I_{X,f}))$. From $I_X \subset I_{X,f}$ we get $\Delta(I_{X,f}) \subset \Delta(I_X)$, in particular $\Delta(I_{X,f})$ is finite which implies $\#(V(I_{X,f})) \leq \#(\Delta(I_{X,f}))$ and we get $N - \#(V(I_{X,f})) \geq N - \#(\Delta(I_{X,f}))$. From Buchberger's result we can assume that f is a linear combination of monomials in $\Delta(I_X)$ so that $\text{lm}(f) \in \Delta(I_X)$. It is not difficult to prove that $N - \#(\Delta(I_{X,f})) \geq \{M \in \Delta(I_X) \mid \text{lm}(f) | M\}$, so the idea now is to determine for each monomial $M' \in \Delta(I_X)$ the cardinality of the set $\{M \in \Delta(I_X) \mid M' | M\}$, and from this determine a lower bound for $d_{\min}(C_L)$. Moreover, it is true that if $\{g_1, \ldots, g_s\}$ is a Gröbner basis for I_X and $\{f, g_1, \ldots, g_s\}$ is a Gröbner basis for $I_{X,f}$ then this bound (with $M' = \text{lm}(f)$) is the true value of the minimum distance.

These techniques have been used to determine the parameters of codes C_{L_d} when $X = A_1 \times \cdots \times A_N \in \mathbb{A}^N(\mathbb{F}_q)$, as well as some higher Hamming weights for these codes.

Gröbner basis methods and the parameters of C_L

As for the minimum distance $d_{\min}(C_L)$, we would like to estimate the number of zero entries in $\varphi(f + I_X) = (f(P_1), \ldots, f(P_N))$. Let $I_{X,f} := I_X + (f)$, we want to estimate $N - \#(V(I_{X,f}))$. From $I_X \subset I_{X,f}$ we get $\Delta(I_{X,f}) \subset \Delta(I_X)$, in particular $\Delta(I_{X,f})$ is finite which implies $\#(V(I_{X,f})) \leq \#(\Delta(I_{X,f}))$ and we get $N - \#(V(I_{X,f})) \geq N - \#(\Delta(I_{X,f}))$.

From Buchberger's result we can assume that f is a linear combination of monomials in $\Delta(I_X)$ so that $\mathrm{lm}(f) \in \Delta(I_X)$. It is not difficult to prove that $N - \#(\Delta(I_{X,f})) \geq \{M \in \Delta(I_X) \mid \mathrm{lm}(f) | M\}$, so the idea now is to determine for each monomial $M' \in \Delta(I_X)$ the cardinality of the set $\{M \in \Delta(I_X) \mid M' | M\}$, and from this determine a lower bound for $d_{\min}(C_L)$. Moreover, it is true that if $\{g_1, \ldots, g_s\}$ is a Gröbner basis for I_X and $\{f, g_1, \ldots, g_s\}$ is a Gröbner basis for $I_{X,f}$ then this bound (with $M' = \mathrm{lm}(f)$) is the true value of the minimum distance.

These techniques have been used to determine the parameters of codes C_{L_d} when $X = A_1 \times \cdots \times A_N \in \mathbb{A}^N(\mathbb{F}_q)$, as well as some higher Hamming weights for these codes.

Gröbner basis methods and the parameters of C_L

As for the minimum distance $d_{\min}(C_L)$, we would like to estimate the number of zero entries in $\varphi(f + I_X) = (f(P_1), \ldots, f(P_N))$. Let $I_{X,f} := I_X + (f)$, we want to estimate $N - \#(V(I_{X,f}))$. From $I_X \subset I_{X,f}$ we get $\Delta(I_{X,f}) \subset \Delta(I_X)$, in particular $\Delta(I_{X,f})$ is finite which implies $\#(V(I_{X,f})) \leq \#(\Delta(I_{X,f}))$ and we get $N - \#(V(I_{X,f})) \geq N - \#(\Delta(I_{X,f}))$.

From Buchberger's result we can assume that f is a linear combination of monomials in $\Delta(I_X)$ so that $\mathrm{lm}(f) \in \Delta(I_X)$. It is not difficult to prove that $N - \#(\Delta(I_{X,f})) \geq \{M \in \Delta(I_X) \mid \mathrm{lm}(f) | M\}$, so the idea now is to determine for each monomial $M' \in \Delta(I_X)$ the cardinality of the set $\{M \in \Delta(I_X) \mid M' | M\}$, and from this determine a lower bound for $d_{\min}(C_L)$. Moreover, it is true that if $\{g_1, \ldots, g_s\}$ is a Gröbner basis for I_X and $\{f, g_1, \ldots, g_s\}$ is a Gröbner basis for $I_{X,f}$ then this bound (with $M' = \mathrm{lm}(f)$) is the true value of the minimum distance.

These techniques have been used to determine the parameters of codes C_{L_d} when $X = A_1 \times \cdots \times A_N \in \mathbb{A}^N(\mathbb{F}_q)$, as well as some higher Hamming weights for these codes.

Gröbner basis methods and the parameters of C_L

As for the minimum distance $d_{\min}(C_L)$, we would like to estimate the number of zero entries in $\varphi(f + I_X) = (f(P_1), \ldots, f(P_N))$. Let $I_{X,f} := I_X + (f)$, we want to estimate $N - \#(V(I_{X,f}))$. From $I_X \subset I_{X,f}$ we get $\Delta(I_{X,f}) \subset \Delta(I_X)$, in particular $\Delta(I_{X,f})$ is finite which implies $\#(V(I_{X,f})) \leq \#(\Delta(I_{X,f}))$ and we get $N - \#(V(I_{X,f})) \geq N - \#(\Delta(I_{X,f}))$.

From Buchberger's result we can assume that f is a linear combination of monomials in $\Delta(I_X)$ so that $\text{lm}(f) \in \Delta(I_X)$. It is not difficult to prove that $N - \#(\Delta(I_{X,f})) \geq \{M \in \Delta(I_X) \mid \text{lm}(f) \mid M\}$, so the idea now is to determine for each monomial $M' \in \Delta(I_X)$ the cardinality of the set $\{M \in \Delta(I_X) \mid M' \mid M\}$, and from this determine a lower bound for $d_{\min}(C_L)$. Moreover, it is true that if $\{g_1, \ldots, g_s\}$ is a Gröbner basis for I_X and $\{f, g_1, \ldots, g_s\}$ is a Gröbner basis for $I_{X,f}$ then this bound (with $M' = \text{lm}(f)$) is the true value of the minimum distance.

These techniques have been used to determine the parameters of codes C_{L_d} when $X = A_1 \times \cdots \times A_N \in \mathbb{A}^N(\mathbb{F}_q)$, as well as some higher Hamming weights for these codes.

Gröbner basis methods and the parameters of C_L

As for the minimum distance $d_{\min}(C_L)$, we would like to estimate the number of zero entries in $\varphi(f + I_X) = (f(P_1), \ldots, f(P_N))$. Let $I_{X,f} := I_X + (f)$, we want to estimate $N - \#(V(I_{X,f}))$. From $I_X \subset I_{X,f}$ we get $\Delta(I_{X,f}) \subset \Delta(I_X)$, in particular $\Delta(I_{X,f})$ is finite which implies $\#(V(I_{X,f})) \leq \#(\Delta(I_{X,f}))$ and we get $N - \#(V(I_{X,f})) \geq N - \#(\Delta(I_{X,f}))$.

From Buchberger's result we can assume that f is a linear combination of monomials in $\Delta(I_X)$ so that $\mathrm{lm}(f) \in \Delta(I_X)$. It is not difficult to prove that $N - \#(\Delta(I_{X,f})) \geq \{M \in \Delta(I_X) \mid \mathrm{lm}(f) | M\}$, so the idea now is to determine for each monomial $M' \in \Delta(I_X)$ the cardinality of the set $\{M \in \Delta(I_X) \mid M' | M\}$, and from this determine a lower bound for $d_{\min}(C_L)$. Moreover, it is true that if $\{g_1, \ldots, g_s\}$ is a Gröbner basis for I_X and $\{f, g_1, \ldots, g_s\}$ is a Gröbner basis for $I_{X,f}$ then this bound (with $M' = \mathrm{lm}(f)$) is the true value of the minimum distance.

These techniques have been used to determine the parameters of codes C_{L_d} when $X = A_1 \times \cdots \times A_N \in \mathbb{A}^N(\mathbb{F}_q)$, as well as some higher Hamming weights for these codes.

Gröbner basis methods and the parameters of C_L

As for the minimum distance $d_{\min}(C_L)$, we would like to estimate the number of zero entries in $\varphi(f + I_X) = (f(P_1), \ldots, f(P_N))$. Let $I_{X,f} := I_X + (f)$, we want to estimate $N - \#(V(I_{X,f}))$. From $I_X \subset I_{X,f}$ we get $\Delta(I_{X,f}) \subset \Delta(I_X)$, in particular $\Delta(I_{X,f})$ is finite which implies $\#(V(I_{X,f})) \leq \#(\Delta(I_{X,f}))$ and we get $N - \#(V(I_{X,f})) \geq N - \#(\Delta(I_{X,f}))$.

From Buchberger's result we can assume that f is a linear combination of monomials in $\Delta(I_X)$ so that $\mathrm{lm}(f) \in \Delta(I_X)$. It is not difficult to prove that $N - \#(\Delta(I_{X,f})) \geq \{M \in \Delta(I_X) \mid \mathrm{lm}(f) | M\}$, so the idea now is to determine for each monomial $M' \in \Delta(I_X)$ the cardinality of the set $\{M \in \Delta(I_X) \mid M' | M\}$, and from this determine a lower bound for $d_{\min}(C_L)$. Moreover, it is true that if $\{g_1, \ldots, g_s\}$ is a Gröbner basis for I_X and $\{f, g_1, \ldots, g_s\}$ is a Gröbner basis for $I_{X,f}$ then this bound (with $M' = \mathrm{lm}(f)$) is the true value of the minimum distance.

These techniques have been used to determine the parameters of codes C_{L_d} when $X = A_1 \times \cdots \times A_N \in \mathbb{A}^N(\mathbb{F}_q)$, as well as some higher Hamming weights for these codes.

Gröbner basis methods and the parameters of C_L

As for the minimum distance $d_{\min}(C_L)$, we would like to estimate the number of zero entries in $\varphi(f + I_X) = (f(P_1), \ldots, f(P_N))$. Let $I_{X,f} := I_X + (f)$, we want to estimate $N - \#(V(I_{X,f}))$. From $I_X \subset I_{X,f}$ we get $\Delta(I_{X,f}) \subset \Delta(I_X)$, in particular $\Delta(I_{X,f})$ is finite which implies $\#(V(I_{X,f})) \leq \#(\Delta(I_{X,f}))$ and we get $N - \#(V(I_{X,f})) \geq N - \#(\Delta(I_{X,f}))$.

From Buchberger's result we can assume that f is a linear combination of monomials in $\Delta(I_X)$ so that $\mathrm{lm}(f) \in \Delta(I_X)$. It is not difficult to prove that $N - \#(\Delta(I_{X,f})) \geq \{M \in \Delta(I_X) \mid \mathrm{lm}(f) | M\}$, so the idea now is to determine for each monomial $M' \in \Delta(I_X)$ the cardinality of the set $\{M \in \Delta(I_X) \mid M' | M\}$, and from this determine a lower bound for $d_{\min}(C_L)$. Moreover, it is true that if $\{g_1, \ldots, g_s\}$ is a Gröbner basis for I_X and $\{f, g_1, \ldots, g_s\}$ is a Gröbner basis for $I_{X,f}$ then this bound (with $M' = \mathrm{lm}(f)$) is the true value of the minimum distance.

These techniques have been used to determine the parameters of codes C_{L_d} when $X = A_1 \times \cdots \times A_N \in \mathbb{A}^N(\mathbb{F}_q)$, as well as some higher Hamming weights for these codes.

Gröbner basis methods and the parameters of C_L

As for the minimum distance $d_{\min}(C_L)$, we would like to estimate the number of zero entries in $\varphi(f + I_X) = (f(P_1), \ldots, f(P_N))$. Let $I_{X,f} := I_X + (f)$, we want to estimate $N - \#(V(I_{X,f}))$. From $I_X \subset I_{X,f}$ we get $\Delta(I_{X,f}) \subset \Delta(I_X)$, in particular $\Delta(I_{X,f})$ is finite which implies $\#(V(I_{X,f})) \leq \#(\Delta(I_{X,f}))$ and we get $N - \#(V(I_{X,f})) \geq N - \#(\Delta(I_{X,f}))$.

From Buchberger's result we can assume that f is a linear combination of monomials in $\Delta(I_X)$ so that $\mathrm{lm}(f) \in \Delta(I_X)$. It is not difficult to prove that $N - \#(\Delta(I_{X,f})) \geq \{M \in \Delta(I_X) \mid \mathrm{lm}(f) | M\}$, so the idea now is to determine for each monomial $M' \in \Delta(I_X)$ the cardinality of the set $\{M \in \Delta(I_X) \mid M' | M\}$, and from this determine a lower bound for $d_{\min}(C_L)$. Moreover, it is true that if $\{g_1, \ldots, g_s\}$ is a Gröbner basis for I_X and $\{f, g_1, \ldots, g_s\}$ is a Gröbner basis for $I_{X,f}$ then this bound (with $M' = \mathrm{lm}(f)$) is the true value of the minimum distance.

These techniques have been used to determine the parameters of codes C_{L_d} when $X = A_1 \times \cdots \times A_N \in \mathbb{A}^N(\mathbb{F}_q)$, as well as some higher Hamming weights for these codes.

Gröbner basis methods and the parameters of C_L

As for the minimum distance $d_{\min}(C_L)$, we would like to estimate the number of zero entries in $\varphi(f + I_X) = (f(P_1), \ldots, f(P_N))$. Let $I_{X,f} := I_X + (f)$, we want to estimate $N - \#(V(I_{X,f}))$. From $I_X \subset I_{X,f}$ we get $\Delta(I_{X,f}) \subset \Delta(I_X)$, in particular $\Delta(I_{X,f})$ is finite which implies $\#(V(I_{X,f})) \leq \#(\Delta(I_{X,f}))$ and we get $N - \#(V(I_{X,f})) \geq N - \#(\Delta(I_{X,f}))$.

From Buchberger's result we can assume that f is a linear combination of monomials in $\Delta(I_X)$ so that $\mathrm{lm}(f) \in \Delta(I_X)$. It is not difficult to prove that $N - \#(\Delta(I_{X,f})) \geq \{M \in \Delta(I_X) \mid \mathrm{lm}(f) | M\}$, so the idea now is to determine for each monomial $M' \in \Delta(I_X)$ the cardinality of the set $\{M \in \Delta(I_X) \mid M' | M\}$, and from this determine a lower bound for $d_{\min}(C_L)$. Moreover, it is true that if $\{g_1, \ldots, g_s\}$ is a Gröbner basis for I_X and $\{f, g_1, \ldots, g_s\}$ is a Gröbner basis for $I_{X,f}$ then this bound (with $M' = \mathrm{lm}(f)$) is the true value of the minimum distance.

These techniques have been used to determine the parameters of codes C_{L_d} when $X = A_1 \times \cdots \times A_N \in \mathbb{A}^N(\mathbb{F}_q)$, as well as some higher Hamming weights for these codes.

Gröbner basis methods and the parameters of C_L

As for the minimum distance $d_{\min}(C_L)$, we would like to estimate the number of zero entries in $\varphi(f + I_X) = (f(P_1), \ldots, f(P_N))$. Let $I_{X,f} := I_X + (f)$, we want to estimate $N - \#(V(I_{X,f}))$. From $I_X \subset I_{X,f}$ we get $\Delta(I_{X,f}) \subset \Delta(I_X)$, in particular $\Delta(I_{X,f})$ is finite which implies $\#(V(I_{X,f})) \leq \#(\Delta(I_{X,f}))$ and we get $N - \#(V(I_{X,f})) \geq N - \#(\Delta(I_{X,f}))$.

From Buchberger's result we can assume that f is a linear combination of monomials in $\Delta(I_X)$ so that $\mathrm{lm}(f) \in \Delta(I_X)$. It is not difficult to prove that $N - \#(\Delta(I_{X,f})) \geq \{M \in \Delta(I_X) \mid \mathrm{lm}(f) | M\}$, so the idea now is to determine for each monomial $M' \in \Delta(I_X)$ the cardinality of the set $\{M \in \Delta(I_X) \mid M' | M\}$, and from this determine a lower bound for $d_{\min}(C_L)$. Moreover, it is true that if $\{g_1, \ldots, g_s\}$ is a Gröbner basis for I_X and $\{f, g_1, \ldots, g_s\}$ is a Gröbner basis for $I_{X,f}$ then this bound (with $M' = \mathrm{lm}(f)$) is the true value of the minimum distance.

These techniques have been used to determine the parameters of codes C_{L_d} when $X = A_1 \times \cdots \times A_N \in \mathbb{A}^N(\mathbb{F}_q)$, as well as some higher Hamming weights for these codes.

Gröbner basis methods and the parameters of C_L

As for the minimum distance $d_{\min}(C_L)$, we would like to estimate the number of zero entries in $\varphi(f + I_X) = (f(P_1), \ldots, f(P_N))$. Let $I_{X,f} := I_X + (f)$, we want to estimate $N - \#(V(I_{X,f}))$. From $I_X \subset I_{X,f}$ we get $\Delta(I_{X,f}) \subset \Delta(I_X)$, in particular $\Delta(I_{X,f})$ is finite which implies $\#(V(I_{X,f})) \leq \#(\Delta(I_{X,f}))$ and we get $N - \#(V(I_{X,f})) \geq N - \#(\Delta(I_{X,f}))$.

From Buchberger's result we can assume that f is a linear combination of monomials in $\Delta(I_X)$ so that $\mathrm{lm}(f) \in \Delta(I_X)$. It is not difficult to prove that $N - \#(\Delta(I_{X,f})) \geq \{M \in \Delta(I_X) \mid \mathrm{lm}(f) | M\}$, so the idea now is to determine for each monomial $M' \in \Delta(I_X)$ the cardinality of the set $\{M \in \Delta(I_X) \mid M' | M\}$, and from this determine a lower bound for $d_{\min}(C_L)$. Moreover, it is true that if $\{g_1, \ldots, g_s\}$ is a Gröbner basis for I_X and $\{f, g_1, \ldots, g_s\}$ is a Gröbner basis for $I_{X,f}$ then this bound (with $M' = \mathrm{lm}(f)$) is the true value of the minimum distance.

These techniques have been used to determine the parameters of codes C_{L_d} when $X = A_1 \times \cdots \times A_N \in \mathbb{A}^N(\mathbb{F}_q)$, as well as some higher Hamming weights for these codes.

Gröbner basis methods and the parameters of C_L

As for the minimum distance $d_{\min}(C_L)$, we would like to estimate the number of zero entries in $\varphi(f + I_X) = (f(P_1), \ldots, f(P_N))$. Let $I_{X,f} := I_X + (f)$, we want to estimate $N - \#(V(I_{X,f}))$. From $I_X \subset I_{X,f}$ we get $\Delta(I_{X,f}) \subset \Delta(I_X)$, in particular $\Delta(I_{X,f})$ is finite which implies $\#(V(I_{X,f})) \leq \#(\Delta(I_{X,f}))$ and we get $N - \#(V(I_{X,f})) \geq N - \#(\Delta(I_{X,f}))$.

From Buchberger's result we can assume that f is a linear combination of monomials in $\Delta(I_X)$ so that $\text{lm}(f) \in \Delta(I_X)$. It is not difficult to prove that $N - \#(\Delta(I_{X,f})) \geq \{M \in \Delta(I_X) \mid \text{lm}(f) | M\}$, so the idea now is to determine for each monomial $M' \in \Delta(I_X)$ the cardinality of the set $\{M \in \Delta(I_X) \mid M' | M\}$, and from this determine a lower bound for $d_{\min}(C_L)$. Moreover, it is true that if $\{g_1, \ldots, g_s\}$ is a Gröbner basis for I_X and $\{f, g_1, \ldots, g_s\}$ is a Gröbner basis for $I_{X,f}$ then this bound (with $M' = \text{lm}(f)$) is the true value of the minimum distance.

These techniques have been used to determine the parameters of codes C_{L_d} when $X = A_1 \times \cdots \times A_N \in \mathbb{A}^N(\mathbb{F}_q)$, as well as some higher Hamming weights for these codes.

Codes defined over projective varieties

One may also use a set $S = \{P_1, \ldots, P_N\} \subset \mathbb{P}^\ell(\mathbb{F}_q)$ to construct a code in the following way. Let $I_S \subset \mathbb{F}_q[X_0, \ldots, X_\ell] = \mathbb{F}_q[\mathbf{X}]$ be the (homogeneous) ideal of S . We know that $\mathbb{F}_q[\mathbf{X}]/I_S = \bigoplus_{d=0}^{\infty} \mathbb{F}_q[\mathbf{X}]_d/I_S(d)$, then fix a d consider the evaluation morphism $\varphi : \mathbb{F}_q[\mathbf{X}]_d/I_S(d) \to \mathbb{F}_q^N$ where $\varphi(f + I_X) = (f(P_1), \ldots, f(P_N))$ (with the points written in "standard notation"). Observe that φ is injective, so defining $C_d = \mathrm{Im}(\varphi)$ we have $\dim(C_d) = \dim_{\mathbb{F}_q} \mathbb{F}_q[\mathbf{X}]_d/I_S(d)$.

In a joint work with Victor G.L. Neumann we chose S to be the set of points on a rational normal scroll and used some ideas from the affine case to calculate the parameters of C_d

Let m and n be integers such that $1 \leq m \leq n$ and let $\ell = n + m + 1$. A rational normal scroll is the algebraic surface defined by

$$S = \left\{ (x_0 : \cdots : x_\ell) \in \mathbb{P}^\ell(\mathbb{F}_q) \mid rank \begin{pmatrix} x_0 & \cdots & x_{n-1} & x_{n+1} & \cdots & x_{\ell-1} \\ x_1 & \cdots & x_n & x_{n+2} & \cdots & x_\ell \end{pmatrix} = 1 \right\}.$$

Codes defined over projective varieties

One may also use a set $S = \{P_1, \ldots, P_N\} \subset \mathbb{P}^\ell(\mathbb{F}_q)$ to construct a code in the following way. Let $I_S \subset \mathbb{F}_q[X_0, \ldots, X_\ell] = \mathbb{F}_q[\mathbf{X}]$ be the (homogeneous) ideal of S . We know that $\mathbb{F}_q[\mathbf{X}]/I_S = \bigoplus_{d=0}^{\infty} \mathbb{F}_q[\mathbf{X}]_d/I_S(d)$, then fix a d consider the evaluation morphism $\varphi : \mathbb{F}_q[\mathbf{X}]_d/I_S(d) \to \mathbb{F}_q^N$ where $\varphi(f + I_X) = (f(P_1), \ldots, f(P_N))$ (with the points written in "standard notation"). Observe that φ is injective, so defining $C_d = \mathrm{Im}(\varphi)$ we have $\dim(C_d) = \dim_{\mathbb{F}_q} \mathbb{F}_q[\mathbf{X}]_d/I_S(d)$.

In a joint work with Victor G.L. Neumann we chose S to be the set of points on a rational normal scroll and used some ideas from the affine case to calculate the parameters of C_d

Let m and n be integers such that $1 \leq m \leq n$ and let $\ell = n + m + 1$. A rational normal scroll is the algebraic surface defined by

$$S = \left\{ (x_0 : \cdots : x_\ell) \in \mathbb{P}^\ell(\mathbb{F}_q) \mid rank \begin{pmatrix} x_0 & \cdots & x_{n-1} & x_{n+1} & \cdots & x_{\ell-1} \\ x_1 & \cdots & x_n & x_{n+2} & \cdots & x_\ell \end{pmatrix} = 1 \right\}.$$

Codes defined over projective varieties

One may also use a set $S = \{P_1, \dots, P_N\} \subset \mathbb{P}^\ell(\mathbb{F}_q)$ to construct a code in the following way. Let $I_S \subset \mathbb{F}_q[X_0, \dots, X_\ell] = \mathbb{F}_q[\mathbf{X}]$ be the (homogeneous) ideal of S . We know that $\mathbb{F}_q[\mathbf{X}]/I_S = \bigoplus_{d=0}^{\infty} \mathbb{F}_q[\mathbf{X}]_d/I_S(d)$, then fix a d consider the evaluation morphism $\varphi : \mathbb{F}_q[\mathbf{X}]_d/I_S(d) \to \mathbb{F}_q^N$ where $\varphi(f + I_X) = (f(P_1), \dots, f(P_N))$ (with the points written in "standard notation"). Observe that φ is injective, so defining $C_d = \mathrm{Im}(\varphi)$ we have $\dim(C_d) = \dim_{\mathbb{F}_q} \mathbb{F}_q[\mathbf{X}]_d/I_S(d)$.

In a joint work with Victor G.L. Neumann we chose S to be the set of points on a rational normal scroll and used some ideas from the affine case to calculate the parameters of C_d

Let m and n be integers such that $1 \le m \le n$ and let $\ell = n + m + 1$. A rational normal scroll is the algebraic surface defined by

$$S = \left\{ (x_0 : \cdots : x_\ell) \in \mathbb{P}^\ell(\mathbb{F}_q) \mid rank \begin{pmatrix} x_0 & \cdots & x_{n-1} & x_{n+1} & \cdots & x_{\ell-1} \\ x_1 & \cdots & x_n & x_{n+2} & \cdots & x_\ell \end{pmatrix} = 1 \right\}.$$

Codes defined over projective varieties

One may also use a set $S = \{P_1, \ldots, P_N\} \subset \mathbb{P}^\ell(\mathbb{F}_q)$ to construct a code in the following way. Let $I_S \subset \mathbb{F}_q[X_0, \ldots, X_\ell] = \mathbb{F}_q[\mathbf{X}]$ be the (homogeneous) ideal of S . We know that $\mathbb{F}_q[\mathbf{X}]/I_S = \bigoplus_{d=0}^{\infty} \mathbb{F}_q[\mathbf{X}]_d/I_S(d)$, then fix a d consider the evaluation morphism $\varphi : \mathbb{F}_q[\mathbf{X}]_d/I_S(d) \to \mathbb{F}_q^N$ where $\varphi(f + I_X) = (f(P_1), \ldots, f(P_N))$ (with the points written in "standard notation"). Observe that φ is injective, so defining $C_d = \mathrm{Im}(\varphi)$ we have $\dim(C_d) = \dim_{\mathbb{F}_q} \mathbb{F}_q[\mathbf{X}]_d/I_S(d)$.

In a joint work with Victor G.L. Neumann we chose S to be the set of points on a rational normal scroll and used some ideas from the affine case to calculate the parameters of C_d

Let m and n be integers such that $1 \leq m \leq n$ and let $\ell = n + m + 1$. A rational normal scroll is the algebraic surface defined by

$$S = \left\{ (x_0 : \cdots : x_\ell) \in \mathbb{P}^\ell(\mathbb{F}_q) \mid rank \begin{pmatrix} x_0 & \cdots & x_{n-1} & x_{n+1} & \cdots & x_{\ell-1} \\ x_1 & \cdots & x_n & x_{n+2} & \cdots & x_\ell \end{pmatrix} = 1 \right\}.$$

Codes defined over projective varieties

One may also use a set $S = \{P_1, \dots, P_N\} \subset \mathbb{P}^\ell(\mathbb{F}_q)$ to construct a code in the following way. Let $I_S \subset \mathbb{F}_q[X_0, \dots, X_\ell] = \mathbb{F}_q[\mathbf{X}]$ be the (homogeneous) ideal of S . We know that $\mathbb{F}_q[\mathbf{X}]/I_S = \bigoplus_{d=0}^{\infty} \mathbb{F}_q[\mathbf{X}]_d/I_S(d)$, then fix a d consider the evaluation morphism $\varphi : \mathbb{F}_q[\mathbf{X}]_d/I_S(d) \to \mathbb{F}_q^N$ where $\varphi(f + I_X) = (f(P_1), \dots, f(P_N))$ (with the points written in "standard notation"). Observe that φ is injective, so defining $C_d = \mathrm{Im}(\varphi)$ we have $\dim(C_d) = \dim_{\mathbb{F}_q} \mathbb{F}_q[\mathbf{X}]_d/I_S(d)$.

In a joint work with Victor G.L. Neumann we chose S to be the set of points on a rational normal scroll and used some ideas from the affine case to calculate the parameters of C_d

Let m and n be integers such that $1 \leq m \leq n$ and let $\ell = n + m + 1$. A rational normal scroll is the algebraic surface defined by

$$S = \left\{ (x_0 : \cdots : x_\ell) \in \mathbb{P}^\ell(\mathbb{F}_q) \mid rank \begin{pmatrix} x_0 & \cdots & x_{n-1} & x_{n+1} & \cdots & x_{\ell-1} \\ x_1 & \cdots & x_n & x_{n+2} & \cdots & x_\ell \end{pmatrix} = 1 \right\}.$$

Codes defined over projective varieties

One may also use a set $S = \{P_1, \ldots, P_N\} \subset \mathbb{P}^\ell(\mathbb{F}_q)$ to construct a code in the following way. Let $I_S \subset \mathbb{F}_q[X_0, \ldots, X_\ell] = \mathbb{F}_q[\mathbf{X}]$ be the (homogeneous) ideal of S . We know that $\mathbb{F}_q[\mathbf{X}]/I_S = \bigoplus_{d=0}^{\infty} \mathbb{F}_q[\mathbf{X}]_d/I_S(d)$, then fix a d consider the evaluation morphism $\varphi : \mathbb{F}_q[\mathbf{X}]_d/I_S(d) \to \mathbb{F}_q^N$ where $\varphi(f + I_X) = (f(P_1), \ldots, f(P_N))$ (with the points written in "standard notation"). Observe that φ is injective, so defining $C_d = \mathrm{Im}(\varphi)$ we have $\dim(C_d) = \dim_{\mathbb{F}_q} \mathbb{F}_q[\mathbf{X}]_d/I_S(d)$.

In a joint work with Victor G.L. Neumann we chose S to be the set of points on a rational normal scroll and used some ideas from the affine case to calculate the parameters of C_d

Let m and n be integers such that $1 \leq m \leq n$ and let $\ell = n + m + 1$. A rational normal scroll is the algebraic surface defined by

$$S = \left\{ (x_0 : \cdots : x_\ell) \in \mathbb{P}^\ell(\mathbb{F}_q) \mid rank \begin{pmatrix} x_0 & \cdots & x_{n-1} & x_{n+1} & \cdots & x_{\ell-1} \\ x_1 & \cdots & x_n & x_{n+2} & \cdots & x_\ell \end{pmatrix} = 1 \right\}.$$

Codes defined over projective varieties

One may also use a set $S = \{P_1, \ldots, P_N\} \subset \mathbb{P}^\ell(\mathbb{F}_q)$ to construct a code in the following way. Let $I_S \subset \mathbb{F}_q[X_0, \ldots, X_\ell] = \mathbb{F}_q[\mathbf{X}]$ be the (homogeneous) ideal of S . We know that $\mathbb{F}_q[\mathbf{X}]/I_S = \bigoplus_{d=0}^{\infty} \mathbb{F}_q[\mathbf{X}]_d/I_S(d)$, then fix a d consider the evaluation morphism $\varphi : \mathbb{F}_q[\mathbf{X}]_d/I_S(d) \to \mathbb{F}_q^N$ where $\varphi(f + I_X) = (f(P_1), \ldots, f(P_N))$ (with the points written in "standard notation"). Observe that φ is injective, so defining $C_d = \mathrm{Im}(\varphi)$ we have $\dim(C_d) = \dim_{\mathbb{F}_q} \mathbb{F}_q[\mathbf{X}]_d/I_S(d)$.

In a joint work with Victor G.L. Neumann we chose S to be the set of points on a rational normal scroll and used some ideas from the affine case to calculate the parameters of C_d

Let m and n be integers such that $1 \leq m \leq n$ and let $\ell = n + m + 1$. A rational normal scroll is the algebraic surface defined by

$$S = \left\{ (x_0 : \cdots : x_\ell) \in \mathbb{P}^\ell(\mathbb{F}_q) \mid rank \begin{pmatrix} x_0 & \cdots & x_{n-1} & x_{n+1} & \cdots & x_{\ell-1} \\ x_1 & \cdots & x_n & x_{n+2} & \cdots & x_\ell \end{pmatrix} = 1 \right\}.$$

Codes defined over projective varieties

One may also use a set $S = \{P_1, \ldots, P_N\} \subset \mathbb{P}^\ell(\mathbb{F}_q)$ to construct a code in the following way. Let $I_S \subset \mathbb{F}_q[X_0, \ldots, X_\ell] = \mathbb{F}_q[\mathbf{X}]$ be the (homogeneous) ideal of S . We know that $\mathbb{F}_q[\mathbf{X}]/I_S = \bigoplus_{d=0}^{\infty} \mathbb{F}_q[\mathbf{X}]_d/I_S(d)$, then fix a d consider the evaluation morphism $\varphi : \mathbb{F}_q[\mathbf{X}]_d/I_S(d) \to \mathbb{F}_q^N$ where $\varphi(f + I_X) = (f(P_1), \ldots, f(P_N))$ (with the points written in "standard notation"). Observe that φ is injective, so defining $C_d = \mathrm{Im}(\varphi)$ we have $\dim(C_d) = \dim_{\mathbb{F}_q} \mathbb{F}_q[\mathbf{X}]_d/I_S(d)$.

In a joint work with Victor G.L. Neumann we chose S to be the set of points on a rational normal scroll and used some ideas from the affine case to calculate the parameters of C_d

Let m and n be integers such that $1 \leq m \leq n$ and let $\ell = n + m + 1$. A rational normal scroll is the algebraic surface defined by

$$S = \left\{ (x_0 : \cdots : x_\ell) \in \mathbb{P}^\ell(\mathbb{F}_q) \mid rank \begin{pmatrix} x_0 & \cdots & x_{n-1} & x_{n+1} & \cdots & x_{\ell-1} \\ x_1 & \cdots & x_n & x_{n+2} & \cdots & x_\ell \end{pmatrix} = 1 \right\}.$$

Codes defined over projective varieties

One may also use a set $S = \{P_1, \ldots, P_N\} \subset \mathbb{P}^\ell(\mathbb{F}_q)$ to construct a code in the following way. Let $I_S \subset \mathbb{F}_q[X_0, \ldots, X_\ell] = \mathbb{F}_q[\mathbf{X}]$ be the (homogeneous) ideal of S . We know that $\mathbb{F}_q[\mathbf{X}]/I_S = \bigoplus_{d=0}^{\infty} \mathbb{F}_q[\mathbf{X}]_d/I_S(d)$, then fix a d consider the evaluation morphism $\varphi : \mathbb{F}_q[\mathbf{X}]_d/I_S(d) \to \mathbb{F}_q^N$ where $\varphi(f + I_X) = (f(P_1), \ldots, f(P_N))$ (with the points written in "standard notation"). Observe that φ is injective, so defining $C_d = \mathrm{Im}(\varphi)$ we have $\dim(C_d) = \dim_{\mathbb{F}_q} \mathbb{F}_q[\mathbf{X}]_d/I_S(d)$.

In a joint work with Victor G.L. Neumann we chose S to be the set of points on a rational normal scroll and used some ideas from the affine case to calculate the parameters of C_d

Let m and n be integers such that $1 \leq m \leq n$ and let $\ell = n + m + 1$. A rational normal scroll is the algebraic surface defined by

$$S = \left\{ (x_0 : \cdots : x_\ell) \in \mathbb{P}^\ell(\mathbb{F}_q) \mid rank \begin{pmatrix} x_0 & \cdots & x_{n-1} & x_{n+1} & \cdots & x_{\ell-1} \\ x_1 & \cdots & x_n & x_{n+2} & \cdots & x_\ell \end{pmatrix} = 1 \right\}.$$

Codes defined over projective varieties

One may also use a set $S = \{P_1, \ldots, P_N\} \subset \mathbb{P}^\ell(\mathbb{F}_q)$ to construct a code in the following way. Let $I_S \subset \mathbb{F}_q[X_0, \ldots, X_\ell] = \mathbb{F}_q[\mathbf{X}]$ be the (homogeneous) ideal of S . We know that $\mathbb{F}_q[\mathbf{X}]/I_S = \bigoplus_{d=0}^{\infty} \mathbb{F}_q[\mathbf{X}]_d / I_S(d)$, then fix a d consider the evaluation morphism $\varphi : \mathbb{F}_q[\mathbf{X}]_d / I_S(d) \to \mathbb{F}_q^N$ where $\varphi(f + I_X) = (f(P_1), \ldots, f(P_N))$ (with the points written in "standard notation"). Observe that φ is injective, so defining $C_d = \mathrm{Im}(\varphi)$ we have $\dim(C_d) = \dim_{\mathbb{F}_q} \mathbb{F}_q[\mathbf{X}]_d / I_S(d)$.

In a joint work with Victor G.L. Neumann we chose S to be the set of points on a rational normal scroll and used some ideas from the affine case to calculate the parameters of C_d

Let m and n be integers such that $1 \leq m \leq n$ and let $\ell = n + m + 1$. A rational normal scroll is the algebraic surface defined by

$$S = \left\{ (x_0 : \cdots : x_\ell) \in \mathbb{P}^\ell(\mathbb{F}_q) \mid rank \begin{pmatrix} x_0 & \cdots & x_{n-1} & x_{n+1} & \cdots & x_{\ell-1} \\ x_1 & \cdots & x_n & x_{n+2} & \cdots & x_\ell \end{pmatrix} = 1 \right\}.$$

Codes defined over projective varieties

One may also use a set $S = \{P_1, \ldots, P_N\} \subset \mathbb{P}^\ell(\mathbb{F}_q)$ to construct a code in the following way. Let $I_S \subset \mathbb{F}_q[X_0, \ldots, X_\ell] = \mathbb{F}_q[\mathbf{X}]$ be the (homogeneous) ideal of S . We know that $\mathbb{F}_q[\mathbf{X}]/I_S = \bigoplus_{d=0}^{\infty} \mathbb{F}_q[\mathbf{X}]_d / I_S(d)$, then fix a d consider the evaluation morphism $\varphi : \mathbb{F}_q[\mathbf{X}]_d / I_S(d) \to \mathbb{F}_q^N$ where $\varphi(f + I_X) = (f(P_1), \ldots, f(P_N))$ (with the points written in "standard notation"). Observe that φ is injective, so defining $C_d = \mathrm{Im}(\varphi)$ we have $\dim(C_d) = \dim_{\mathbb{F}_q} \mathbb{F}_q[\mathbf{X}]_d / I_S(d)$.

In a joint work with Victor G.L. Neumann we chose S to be the set of points on a rational normal scroll and used some ideas from the affine case to calculate the parameters of C_d

Let m and n be integers such that $1 \leq m \leq n$ and let $\ell = n + m + 1$. A rational normal scroll is the algebraic surface defined by

$$S = \left\{ (x_0 : \cdots : x_\ell) \in \mathbb{P}^\ell(\mathbb{F}_q) \mid rank \begin{pmatrix} x_0 & \cdots & x_{n-1} & x_{n+1} & \cdots & x_{\ell-1} \\ x_1 & \cdots & x_n & x_{n+2} & \cdots & x_\ell \end{pmatrix} = 1 \right\}.$$

Codes defined over projective varieties

One may also use a set $S = \{P_1, \ldots, P_N\} \subset \mathbb{P}^\ell(\mathbb{F}_q)$ to construct a code in the following way. Let $I_S \subset \mathbb{F}_q[X_0, \ldots, X_\ell] = \mathbb{F}_q[\mathbf{X}]$ be the (homogeneous) ideal of S . We know that $\mathbb{F}_q[\mathbf{X}]/I_S = \bigoplus_{d=0}^{\infty} \mathbb{F}_q[\mathbf{X}]_d/I_S(d)$, then fix a d consider the evaluation morphism $\varphi : \mathbb{F}_q[\mathbf{X}]_d/I_S(d) \to \mathbb{F}_q^N$ where $\varphi(f + I_X) = (f(P_1), \ldots, f(P_N))$ (with the points written in "standard notation"). Observe that φ is injective, so defining $C_d = \mathrm{Im}(\varphi)$ we have $\dim(C_d) = \dim_{\mathbb{F}_q} \mathbb{F}_q[\mathbf{X}]_d/I_S(d)$.

In a joint work with Victor G.L. Neumann we chose S to be the set of points on a rational normal scroll and used some ideas from the affine case to calculate the parameters of C_d

Let m and n be integers such that $1 \leq m \leq n$ and let $\ell = n + m + 1$. A rational normal scroll is the algebraic surface defined by

$$S = \left\{ (x_0 : \cdots : x_\ell) \in \mathbb{P}^\ell(\mathbb{F}_q) \mid rank \begin{pmatrix} x_0 & \cdots & x_{n-1} & x_{n+1} & \cdots & x_{\ell-1} \\ x_1 & \cdots & x_n & x_{n+2} & \cdots & x_\ell \end{pmatrix} = 1 \right\}.$$

Codes defined over projective varieties

One may also use a set $S = \{P_1, \ldots, P_N\} \subset \mathbb{P}^\ell(\mathbb{F}_q)$ to construct a code in the following way. Let $I_S \subset \mathbb{F}_q[X_0, \ldots, X_\ell] = \mathbb{F}_q[\mathbf{X}]$ be the (homogeneous) ideal of S . We know that $\mathbb{F}_q[\mathbf{X}]/I_S = \bigoplus_{d=0}^{\infty} \mathbb{F}_q[\mathbf{X}]_d/I_S(d)$, then fix a d consider the evaluation morphism $\varphi : \mathbb{F}_q[\mathbf{X}]_d/I_S(d) \to \mathbb{F}_q^N$ where $\varphi(f + I_X) = (f(P_1), \ldots, f(P_N))$ (with the points written in "standard notation"). Observe that φ is injective, so defining $C_d = \mathrm{Im}(\varphi)$ we have $\dim(C_d) = \dim_{\mathbb{F}_q} \mathbb{F}_q[\mathbf{X}]_d/I_S(d)$.

In a joint work with Victor G.L. Neumann we chose S to be the set of points on a rational normal scroll and used some ideas from the affine case to calculate the parameters of C_d

Let m and n be integers such that $1 \leq m \leq n$ and let $\ell = n + m + 1$. A rational normal scroll is the algebraic surface defined by

$$S = \left\{ (x_0 : \cdots : x_\ell) \in \mathbb{P}^\ell(\mathbb{F}_q) \mid rank \begin{pmatrix} x_0 & \cdots & x_{n-1} & x_{n+1} & \cdots & x_{\ell-1} \\ x_1 & \cdots & x_n & x_{n+2} & \cdots & x_\ell \end{pmatrix} = 1 \right\}.$$

Codes defined on rational normal scrolls

It is not difficult to prove that $S = S_0 \cup C_\infty \cup L_0$, where
$S_0 = \{(1 : a : \cdots : a^n : b : ab : \cdots : a^m b) \mid (a, b) \in \mathbb{A}^2(\mathbb{F}_q)\}$,
$C_\infty = \{(0 : \cdots : 0 : a^m : a^{m-1}b : \cdots : b^m) \in S \mid (a : b) \in \mathbb{P}^1(\mathbb{F}_q)\}$, and
$L_0 = \{(0 : \cdots : 0 : a : 0 : \cdots 0 : b) \in S \mid (a : b) \in \mathbb{P}^1(\mathbb{F}_q)\}$.

The union is disjoint, except for the point $(0 : \ldots : 0 : 1) = L_0 \cap C_\infty$, thus S has $N := q^2 + 2(q+1) - 1 = (q+1)^2$ points hence C_d is a code of length $(q+1)^2$. Since $\dim C_d = \dim_{\mathbb{F}_q} \mathbb{F}_q[\mathbf{X}]_d / I_S(d)$ to calculate $\dim C_d$ we could count the number of monomials of degree d in some footprint of I_S, so we look for a description of I_S.

A good place to start is to examine the ideal I generated by the set G of binomials "suggested" by the definition of S, namely
$G = \{X_i X_j - X_{i+1} X_{j-1} \mid 0 \le i \le \ell - 2\,, i \neq n, i+1 < j \le \ell\,, j \neq n+1\}$
It turns out that G is a Gröbner basis for I w.r.t. the graded lexicographic order $\prec$ where $X_\ell \prec \cdots \prec X_0$, but then $I \subsetneq I_S$. Take for example $1 = m < n = 2$ and $q = 5$. We have $X_0 X_2 - X_1^2 \in G$ and if $I = I_S$ then we should have $X_0^5 X_2 - X_1^6 \in I$, but in this case
$X_0^5 X_2 - X_1^6 - X_0^4(X_0 X_2 - X_1^2) = X_0^4 X_1^2 - X_1^6 \in I$ but this is not true (since $X_0^4 X_1^2$ is not a multiple of $X_0 X_2$, $X_0 X_4$ or $X_1 X_4$).

Codes defined on rational normal scrolls

It is not difficult to prove that $S = S_0 \cup C_\infty \cup L_0$, where

$S_0 = \{(1 : a : \cdots : a^n : b : ab : \cdots : a^m b) \mid (a, b) \in \mathbb{A}^2(\mathbb{F}_q)\}$,
$C_\infty = \{(0 : \cdots : 0 : a^m : a^{m-1}b : \cdots : b^m) \in S \mid (a : b) \in \mathbb{P}^1(\mathbb{F}_q)\}$, and
$L_0 = \{(0 : \cdots : 0 : a : 0 : \cdots 0 : b) \in S \mid (a : b) \in \mathbb{P}^1(\mathbb{F}_q)\}$.

The union is disjoint, except for the point $(0 : \ldots : 0 : 1) = L_0 \cap C_\infty$, thus S has $N := q^2 + 2(q+1) - 1 = (q+1)^2$ points hence C_d is a code of length $(q+1)^2$. Since $\dim C_d = \dim_{\mathbb{F}_q} \mathbb{F}_q[\mathbf{X}]_d / I_S(d)$ to calculate $\dim C_d$ we could count the number of monomials of degree d in some footprint of I_S, so we look for a description of I_S.

A good place to start is to examine the ideal I generated by the set G of binomials "suggested" by the definition of S, namely
$G = \{X_i X_j - X_{i+1} X_{j-1} \mid 0 \leq i \leq \ell - 2\,,\, i \neq n, i+1 < j \leq \ell\,,\, j \neq n+1\}$

It turns out that G is a Gröbner basis for I w.r.t. the graded lexicographic order $\prec$ where $X_\ell \prec \cdots \prec X_0$, but then $I \subsetneq I_S$. Take for example $1 = m < n = 2$ and $q = 5$. We have $X_0 X_2 - X_1^2 \in G$ and if $I = I_S$ then we should have $X_0^5 X_2 - X_1^6 \in I$, but in this case $X_0^5 X_2 - X_1^6 - X_0^4(X_0 X_2 - X_1^2) = X_0^4 X_1^2 - X_1^6 \in I$ but this is not true (since $X_0^4 X_1^2$ is not a multiple of $X_0 X_2$, $X_0 X_4$ or $X_1 X_4$).

Codes defined on rational normal scrolls

It is not difficult to prove that $S = S_0 \cup C_\infty \cup L_0$, where
$S_0 = \{(1 : a : \cdots : a^n : b : ab : \cdots : a^m b) \mid (a,b) \in \mathbb{A}^2(\mathbb{F}_q)\}$,
$C_\infty = \{(0 : \cdots : 0 : a^m : a^{m-1}b : \cdots : b^m) \in S \mid (a : b) \in \mathbb{P}^1(\mathbb{F}_q)\}$, and
$L_0 = \{(0 : \cdots : 0 : a : 0 : \cdots 0 : b) \in S \mid (a : b) \in \mathbb{P}^1(\mathbb{F}_q)\}$.

The union is disjoint, except for the point $(0 : \ldots : 0 : 1) = L_0 \cap C_\infty$, thus S has $N := q^2 + 2(q+1) - 1 = (q+1)^2$ points hence C_d is a code of length $(q+1)^2$. Since $\dim C_d = \dim_{\mathbb{F}_q} \mathbb{F}_q[\mathbf{X}]_d / I_S(d)$ to calculate $\dim C_d$ we could count the number of monomials of degree d in some footprint of I_S, so we look for a description of I_S.

A good place to start is to examine the ideal I generated by the set G of binomials "suggested" by the definition of S, namely
$G = \{X_i X_j - X_{i+1} X_{j-1} \mid 0 \leq i \leq \ell - 2\,,\, i \neq n, i+1 < j \leq \ell\,,\, j \neq n+1\}$
It turns out that G is a Gröbner basis for I w.r.t. the graded lexicographic order $\prec$ where $X_\ell \prec \cdots \prec X_0$, but then $I \subsetneq I_S$. Take for example $1 = m < n = 2$ and $q = 5$. We have $X_0 X_2 - X_1^2 \in G$ and if $I = I_S$ then we should have $X_0^5 X_2 - X_1^6 \in I$, but in this case
$X_0^5 X_2 - X_1^6 - X_0^4(X_0 X_2 - X_1^2) = X_0^4 X_1^2 - X_1^6 \in I$ but this is not true (since $X_0^4 X_1^2$ is not a multiple of $X_0 X_2$, $X_0 X_4$ or $X_1 X_4$).

Codes defined on rational normal scrolls

It is not difficult to prove that $S = S_0 \cup C_\infty \cup L_0$, where
$S_0 = \{(1 : a : \cdots : a^n : b : ab : \cdots : a^m b) \mid (a, b) \in \mathbb{A}^2(\mathbb{F}_q)\}$,
$C_\infty = \{(0 : \cdots : 0 : a^m : a^{m-1}b : \cdots : b^m) \in S \mid (a : b) \in \mathbb{P}^1(\mathbb{F}_q)\}$, and
$L_0 = \{(0 : \cdots : 0 : a : 0 : \cdots 0 : b) \in S \mid (a : b) \in \mathbb{P}^1(\mathbb{F}_q)\}$.

The union is disjoint, except for the point $(0 : \ldots : 0 : 1) = L_0 \cap C_\infty$, thus S has $N := q^2 + 2(q + 1) - 1 = (q + 1)^2$ points hence C_d is a code of length $(q + 1)^2$. Since $\dim C_d = \dim_{\mathbb{F}_q} \mathbb{F}_q[\mathbf{X}]_d / I_S(d)$ to calculate $\dim C_d$ we could count the number of monomials of degree d in some footprint of I_S, so we look for a description of I_S.

A good place to start is to examine the ideal I generated by the set G of binomials "suggested" by the definition of S, namely
$G = \{X_i X_j - X_{i+1} X_{j-1} \mid 0 \leq i \leq \ell - 2 \, , i \neq n, i + 1 < j \leq \ell \, , j \neq n + 1\}$
It turns out that G is a Gröbner basis for I w.r.t. the graded lexicographic order $\prec$ where $X_\ell \prec \cdots \prec X_0$, but then $I \subsetneq I_S$. Take for example $1 = m < n = 2$ and $q = 5$. We have $X_0 X_2 - X_1^2 \in G$ and if $I = I_S$ then we should have $X_0^5 X_2 - X_1^6 \in I$, but in this case
$X_0^5 X_2 - X_1^6 - X_0^4 (X_0 X_2 - X_1^2) = X_0^4 X_1^2 - X_1^6 \in I$ but this is not true (since $X_0^4 X_1^2$ is not a multiple of $X_0 X_2$, $X_0 X_4$ or $X_1 X_4$).

Codes defined on rational normal scrolls

It is not difficult to prove that $S = S_0 \cup C_\infty \cup L_0$, where
$S_0 = \{(1 : a : \cdots : a^n : b : ab : \cdots : a^m b) \mid (a, b) \in \mathbb{A}^2(\mathbb{F}_q)\}$,
$C_\infty = \{(0 : \cdots : 0 : a^m : a^{m-1}b : \cdots : b^m) \in S \mid (a : b) \in \mathbb{P}^1(\mathbb{F}_q)\}$, and
$L_0 = \{(0 : \cdots : 0 : a : 0 : \cdots 0 : b) \in S \mid (a : b) \in \mathbb{P}^1(\mathbb{F}_q)\}$.

The union is disjoint, except for the point $(0 : \ldots : 0 : 1) = L_0 \cap C_\infty$, thus S has $N := q^2 + 2(q+1) - 1 = (q+1)^2$ points hence C_d is a code of length $(q+1)^2$. Since $\dim C_d = \dim_{\mathbb{F}_q} \mathbb{F}_q[\mathbf{X}]_d / I_S(d)$ to calculate $\dim C_d$ we could count the number of monomials of degree d in some footprint of I_S, so we look for a description of I_S.

A good place to start is to examine the ideal I generated by the set G of binomials "suggested" by the definition of S, namely
$G = \{X_i X_j - X_{i+1} X_{j-1} \mid 0 \leq i \leq \ell - 2\, , i \neq n, i+1 < j \leq \ell\, , j \neq n+1\}$
It turns out that G is a Gröbner basis for I w.r.t. the graded lexicographic order $\prec$ where $X_\ell \prec \cdots \prec X_0$, but then $I \subsetneq I_S$. Take for example $1 = m < n = 2$ and $q = 5$. We have $X_0 X_2 - X_1^2 \in G$ and if $I = I_S$ then we should have $X_0^5 X_2 - X_1^6 \in I$, but in this case $X_0^5 X_2 - X_1^6 - X_0^4(X_0 X_2 - X_1^2) = X_0^4 X_1^2 - X_1^6 \in I$ but this is not true (since $X_0^4 X_1^2$ is not a multiple of $X_0 X_2$, $X_0 X_4$ or $X_1 X_4$).

Codes defined on rational normal scrolls

It is not difficult to prove that $S = S_0 \cup C_\infty \cup L_0$, where
$S_0 = \{(1 : a : \cdots : a^n : b : ab : \cdots : a^m b) \mid (a, b) \in \mathbb{A}^2(\mathbb{F}_q)\}$,
$C_\infty = \{(0 : \cdots : 0 : a^m : a^{m-1}b : \cdots : b^m) \in S \mid (a : b) \in \mathbb{P}^1(\mathbb{F}_q)\}$, and
$L_0 = \{(0 : \cdots : 0 : a : 0 : \cdots 0 : b) \in S \mid (a : b) \in \mathbb{P}^1(\mathbb{F}_q)\}$.

The union is disjoint, except for the point $(0 : \ldots : 0 : 1) = L_0 \cap C_\infty$, thus S has $N := q^2 + 2(q+1) - 1 = (q+1)^2$ points hence C_d is a code of length $(q+1)^2$. Since $\dim C_d = \dim_{\mathbb{F}_q} \mathbb{F}_q[\mathbf{X}]_d / I_S(d)$ to calculate $\dim C_d$ we could count the number of monomials of degree d in some footprint of I_S, so we look for a description of I_S.

A good place to start is to examine the ideal I generated by the set G of binomials "suggested" by the definition of S, namely
$G = \{X_i X_j - X_{i+1} X_{j-1} \mid 0 \leq i \leq \ell - 2 \, , \, i \neq n, i+1 < j \leq \ell \, , \, j \neq n+1\}$
It turns out that G is a Gröbner basis for I w.r.t. the graded lexicographic order $\prec$ where $X_\ell \prec \cdots \prec X_0$, but then $I \subsetneq I_S$. Take for example $1 = m < n = 2$ and $q = 5$. We have $X_0 X_2 - X_1^2 \in G$ and if $I = I_S$ then we should have $X_0^5 X_2 - X_1^6 \in I$, but in this case
$X_0^5 X_2 - X_1^6 - X_0^4(X_0 X_2 - X_1^2) = X_0^4 X_1^2 - X_1^6 \in I$ but this is not true (since $X_0^4 X_1^2$ is not a multiple of $X_0 X_2$, $X_0 X_4$ or $X_1 X_4$).

Codes defined on rational normal scrolls

It is not difficult to prove that $S = S_0 \cup C_\infty \cup L_0$, where
$S_0 = \{(1 : a : \cdots : a^n : b : ab : \cdots : a^m b) \mid (a, b) \in \mathbb{A}^2(\mathbb{F}_q)\}$,
$C_\infty = \{(0 : \cdots : 0 : a^m : a^{m-1}b : \cdots : b^m) \in S \mid (a : b) \in \mathbb{P}^1(\mathbb{F}_q)\}$, and
$L_0 = \{(0 : \cdots : 0 : a : 0 : \cdots 0 : b) \in S \mid (a : b) \in \mathbb{P}^1(\mathbb{F}_q)\}$.

The union is disjoint, except for the point $(0 : \ldots : 0 : 1) = L_0 \cap C_\infty$, thus S has $N := q^2 + 2(q + 1) - 1 = (q + 1)^2$ points hence C_d is a code of length $(q + 1)^2$. Since $\dim C_d = \dim_{\mathbb{F}_q} \mathbb{F}_q[\mathbf{X}]_d / I_S(d)$ to calculate $\dim C_d$ we could count the number of monomials of degree d in some footprint of I_S, so we look for a description of I_S.

A good place to start is to examine the ideal I generated by the set G of binomials "suggested" by the definition of S, namely
$G = \{X_i X_j - X_{i+1} X_{j-1} \mid 0 \leq i \leq \ell - 2\, ,\, i \neq n, i + 1 < j \leq \ell\, ,\, j \neq n + 1\}$
It turns out that G is a Gröbner basis for I w.r.t. the graded lexicographic order $\prec$ where $X_\ell \prec \cdots \prec X_0$, but then $I \subsetneq I_S$. Take for example $1 = m < n = 2$ and $q = 5$. We have $X_0 X_2 - X_1^2 \in G$ and if $I = I_S$ then we should have $X_0^5 X_2 - X_1^6 \in I$, but in this case
$X_0^5 X_2 - X_1^6 - X_0^4(X_0 X_2 - X_1^2) = X_0^4 X_1^2 - X_1^6 \in I$ but this is not true (since $X_0^4 X_1^2$ is not a multiple of $X_0 X_2$, $X_0 X_4$ or $X_1 X_4$).

Codes defined on rational normal scrolls

It is not difficult to prove that $S = S_0 \cup C_\infty \cup L_0$, where
$S_0 = \{(1 : a : \cdots : a^n : b : ab : \cdots : a^m b) \mid (a, b) \in \mathbb{A}^2(\mathbb{F}_q)\}$,
$C_\infty = \{(0 : \cdots : 0 : a^m : a^{m-1}b : \cdots : b^m) \in S \mid (a : b) \in \mathbb{P}^1(\mathbb{F}_q)\}$, and
$L_0 = \{(0 : \cdots : 0 : a : 0 : \cdots 0 : b) \in S \mid (a : b) \in \mathbb{P}^1(\mathbb{F}_q)\}$.

The union is disjoint, except for the point $(0 : \ldots : 0 : 1) = L_0 \cap C_\infty$, thus S has $N := q^2 + 2(q+1) - 1 = (q+1)^2$ points hence C_d is a code of length $(q+1)^2$. Since $\dim C_d = \dim_{\mathbb{F}_q} \mathbb{F}_q[\mathbf{X}]_d / I_S(d)$ to calculate $\dim C_d$ we could count the number of monomials of degree d in some footprint of I_S, so we look for a description of I_S.

A good place to start is to examine the ideal I generated by the set G of binomials "suggested" by the definition of S, namely
$G = \{X_i X_j - X_{i+1} X_{j-1} \mid 0 \leq i \leq \ell - 2\, , i \neq n, i+1 < j \leq \ell\, , j \neq n+1\}$
It turns out that G is a Gröbner basis for I w.r.t. the graded lexicographic order $\prec$ where $X_\ell \prec \cdots \prec X_0$, but then $I \subsetneq I_S$. Take for example $1 = m < n = 2$ and $q = 5$. We have $X_0X_2 - X_1^2 \in G$ and if $I = I_S$ then we should have $X_0^5 X_2 - X_1^6 \in I$, but in this case $X_0^5 X_2 - X_1^6 - X_0^4(X_0X_2 - X_1^2) = X_0^4 X_1^2 - X_1^6 \in I$ but this is not true (since $X_0^4 X_1^2$ is not a multiple of X_0X_2, X_0X_4 or X_1X_4).

Codes defined on rational normal scrolls

It is not difficult to prove that $S = S_0 \cup C_\infty \cup L_0$, where
$S_0 = \{(1 : a : \cdots : a^n : b : ab : \cdots : a^m b) \mid (a,b) \in \mathbb{A}^2(\mathbb{F}_q)\}$,
$C_\infty = \{(0 : \cdots : 0 : a^m : a^{m-1}b : \cdots : b^m) \in S \mid (a : b) \in \mathbb{P}^1(\mathbb{F}_q)\}$, and
$L_0 = \{(0 : \cdots : 0 : a : 0 : \cdots 0 : b) \in S \mid (a : b) \in \mathbb{P}^1(\mathbb{F}_q)\}$.

The union is disjoint, except for the point $(0 : \ldots : 0 : 1) = L_0 \cap C_\infty$, thus S has $N := q^2 + 2(q+1) - 1 = (q+1)^2$ points hence C_d is a code of length $(q+1)^2$. Since $\dim C_d = \dim_{\mathbb{F}_q} \mathbb{F}_q[\mathbf{X}]_d / I_S(d)$ to calculate $\dim C_d$ we could count the number of monomials of degree d in some footprint of I_S, so we look for a description of I_S.

A good place to start is to examine the ideal I generated by the set G of binomials "suggested" by the definition of S, namely
$G = \{X_i X_j - X_{i+1} X_{j-1} \mid 0 \leq i \leq \ell - 2\,,\, i \neq n, i+1 < j \leq \ell\,,\, j \neq n+1\}$
It turns out that G is a Gröbner basis for I w.r.t. the graded lexicographic order $\prec$ where $X_\ell \prec \cdots \prec X_0$, but then $I \subsetneq I_S$. Take for example $1 = m < n = 2$ and $q = 5$. We have $X_0 X_2 - X_1^2 \in G$ and if $I = I_S$ then we should have $X_0^5 X_2 - X_1^6 \in I$, but in this case $X_0^5 X_2 - X_1^6 - X_0^4(X_0 X_2 - X_1^2) = X_0^4 X_1^2 - X_1^6 \in I$ but this is not true (since $X_0^4 X_1^2$ is not a multiple of $X_0 X_2$, $X_0 X_4$ or $X_1 X_4$).

Codes defined on rational normal scrolls

It is not difficult to prove that $S = S_0 \cup C_\infty \cup L_0$, where
$S_0 = \{(1 : a : \cdots : a^n : b : ab : \cdots : a^m b) \mid (a, b) \in \mathbb{A}^2(\mathbb{F}_q)\}$,
$C_\infty = \{(0 : \cdots : 0 : a^m : a^{m-1}b : \cdots : b^m) \in S \mid (a : b) \in \mathbb{P}^1(\mathbb{F}_q)\}$, and
$L_0 = \{(0 : \cdots : 0 : a : 0 : \cdots 0 : b) \in S \mid (a : b) \in \mathbb{P}^1(\mathbb{F}_q)\}$.

The union is disjoint, except for the point $(0 : \ldots : 0 : 1) = L_0 \cap C_\infty$, thus S has $N := q^2 + 2(q + 1) - 1 = (q + 1)^2$ points hence C_d is a code of length $(q + 1)^2$. Since $\dim C_d = \dim_{\mathbb{F}_q} \mathbb{F}_q[\mathbf{X}]_d / I_S(d)$ to calculate $\dim C_d$ we could count the number of monomials of degree d in some footprint of I_S, so we look for a description of I_S.

A good place to start is to examine the ideal I generated by the set G of binomials "suggested" by the definition of S, namely
$G = \{X_i X_j - X_{i+1} X_{j-1} \mid 0 \leq i \leq \ell - 2\,,\, i \neq n, i + 1 < j \leq \ell\,,\, j \neq n + 1\}$
It turns out that G is a Gröbner basis for I w.r.t. the graded lexicographic order $\prec$ where $X_\ell \prec \cdots \prec X_0$, but then $I \subsetneq I_S$. Take for example $1 = m < n = 2$ and $q = 5$. We have $X_0 X_2 - X_1^2 \in G$ and if $I = I_S$ then we should have $X_0^5 X_2 - X_1^6 \in I$, but in this case
$X_0^5 X_2 - X_1^6 - X_0^4 (X_0 X_2 - X_1^2) = X_0^4 X_1^2 - X_1^6 \in I$ but this is not true (since $X_0^4 X_1^2$ is not a multiple of $X_0 X_2$, $X_0 X_4$ or $X_1 X_4$).

Codes defined on rational normal scrolls

It is not difficult to prove that $S = S_0 \cup C_\infty \cup L_0$, where
$S_0 = \{(1 : a : \cdots : a^n : b : ab : \cdots : a^m b) \mid (a, b) \in \mathbb{A}^2(\mathbb{F}_q)\}$,
$C_\infty = \{(0 : \cdots : 0 : a^m : a^{m-1}b : \cdots : b^m) \in S \mid (a : b) \in \mathbb{P}^1(\mathbb{F}_q)\}$, and
$L_0 = \{(0 : \cdots : 0 : a : 0 : \cdots 0 : b) \in S \mid (a : b) \in \mathbb{P}^1(\mathbb{F}_q)\}$.

The union is disjoint, except for the point $(0 : \ldots : 0 : 1) = L_0 \cap C_\infty$, thus S has $N := q^2 + 2(q + 1) - 1 = (q + 1)^2$ points hence C_d is a code of length $(q + 1)^2$. Since $\dim C_d = \dim_{\mathbb{F}_q} \mathbb{F}_q[\mathbf{X}]_d / I_S(d)$ to calculate $\dim C_d$ we could count the number of monomials of degree d in some footprint of I_S, so we look for a description of I_S.

A good place to start is to examine the ideal I generated by the set G of binomials "suggested" by the definition of S, namely
$G = \{X_i X_j - X_{i+1} X_{j-1} \mid 0 \leq i \leq \ell - 2 \, , \, i \neq n, i + 1 < j \leq \ell \, , \, j \neq n + 1\}$
It turns out that G is a Gröbner basis for I w.r.t. the graded lexicographic order $\prec$ where $X_\ell \prec \cdots \prec X_0$, but then $I \subsetneq I_S$. Take for example $1 = m < n = 2$ and $q = 5$. We have $X_0 X_2 - X_1^2 \in G$ and if $I = I_S$ then we should have $X_0^5 X_2 - X_1^6 \in I$, but in this case
$X_0^5 X_2 - X_1^6 - X_0^4 (X_0 X_2 - X_1^2) = X_0^4 X_1^2 - X_1^6 \in I$ but this is not true (since $X_0^4 X_1^2$ is not a multiple of $X_0 X_2$, $X_0 X_4$ or $X_1 X_4$).

Codes defined on rational normal scrolls

It is not difficult to prove that $S = S_0 \cup C_\infty \cup L_0$, where
$S_0 = \{(1 : a : \cdots : a^n : b : ab : \cdots : a^m b) \mid (a, b) \in \mathbb{A}^2(\mathbb{F}_q)\}$,
$C_\infty = \{(0 : \cdots : 0 : a^m : a^{m-1}b : \cdots : b^m) \in S \mid (a : b) \in \mathbb{P}^1(\mathbb{F}_q)\}$, and
$L_0 = \{(0 : \cdots : 0 : a : 0 : \cdots 0 : b) \in S \mid (a : b) \in \mathbb{P}^1(\mathbb{F}_q)\}$.

The union is disjoint, except for the point $(0 : \ldots : 0 : 1) = L_0 \cap C_\infty$, thus S has $N := q^2 + 2(q + 1) - 1 = (q + 1)^2$ points hence C_d is a code of length $(q + 1)^2$. Since $\dim C_d = \dim_{\mathbb{F}_q} \mathbb{F}_q[\mathbf{X}]_d / I_S(d)$ to calculate $\dim C_d$ we could count the number of monomials of degree d in some footprint of I_S, so we look for a description of I_S.

A good place to start is to examine the ideal I generated by the set G of binomials "suggested" by the definition of S, namely
$G = \{X_i X_j - X_{i+1} X_{j-1} \mid 0 \le i \le \ell - 2\, , i \ne n, i + 1 < j \le \ell\, , j \ne n + 1\}$
It turns out that G is a Gröbner basis for I w.r.t. the graded lexicographic order $\prec$ where $X_\ell \prec \cdots \prec X_0$, but then $I \subsetneq I_S$. Take for example $1 = m < n = 2$ and $q = 5$. We have $X_0 X_2 - X_1^2 \in G$ and if $I = I_S$ then we should have $X_0^5 X_2 - X_1^6 \in I$, but in this case $X_0^5 X_2 - X_1^6 - X_0^4(X_0 X_2 - X_1^2) = X_0^4 X_1^2 - X_1^6 \in I$ but this is not true (since $X_0^4 X_1^2$ is not a multiple of $X_0 X_2$, $X_0 X_4$ or $X_1 X_4$).

Codes defined on rational normal scrolls

It is not difficult to prove that $S = S_0 \cup C_\infty \cup L_0$, where
$S_0 = \{(1 : a : \cdots : a^n : b : ab : \cdots : a^m b) \mid (a, b) \in \mathbb{A}^2(\mathbb{F}_q)\}$,
$C_\infty = \{(0 : \cdots : 0 : a^m : a^{m-1}b : \cdots : b^m) \in S \mid (a : b) \in \mathbb{P}^1(\mathbb{F}_q)\}$, and
$L_0 = \{(0 : \cdots : 0 : a : 0 : \cdots 0 : b) \in S \mid (a : b) \in \mathbb{P}^1(\mathbb{F}_q)\}$.

The union is disjoint, except for the point $(0 : \ldots : 0 : 1) = L_0 \cap C_\infty$, thus S has $N := q^2 + 2(q+1) - 1 = (q+1)^2$ points hence C_d is a code of length $(q+1)^2$. Since $\dim C_d = \dim_{\mathbb{F}_q} \mathbb{F}_q[\mathbf{X}]_d / I_S(d)$ to calculate $\dim C_d$ we could count the number of monomials of degree d in some footprint of I_S, so we look for a description of I_S.

A good place to start is to examine the ideal I generated by the set G of binomials "suggested" by the definition of S, namely
$G = \{X_i X_j - X_{i+1} X_{j-1} \mid 0 \leq i \leq \ell - 2\,,\, i \neq n, i+1 < j \leq \ell\,,\, j \neq n+1\}$

It turns out that G is a Gröbner basis for I w.r.t. the graded lexicographic order $\prec$ where $X_\ell \prec \cdots \prec X_0$, but then $I \subsetneq I_S$. Take for example $1 = m < n = 2$ and $q = 5$. We have $X_0X_2 - X_1^2 \in G$ and if $I = I_S$ then we should have $X_0^5 X_2 - X_1^6 \in I$, but in this case $X_0^5X_2 - X_1^6 - X_0^4(X_0X_2 - X_1^2) = X_0^4X_1^2 - X_1^6 \in I$ but this is not true (since $X_0^4X_1^2$ is not a multiple of X_0X_2, X_0X_4 or X_1X_4).

Codes defined on rational normal scrolls

It is not difficult to prove that $S = S_0 \cup C_\infty \cup L_0$, where
$S_0 = \{(1 : a : \cdots : a^n : b : ab : \cdots : a^m b) \mid (a, b) \in \mathbb{A}^2(\mathbb{F}_q)\}$,
$C_\infty = \{(0 : \cdots : 0 : a^m : a^{m-1}b : \cdots : b^m) \in S \mid (a : b) \in \mathbb{P}^1(\mathbb{F}_q)\}$, and
$L_0 = \{(0 : \cdots : 0 : a : 0 : \cdots 0 : b) \in S \mid (a : b) \in \mathbb{P}^1(\mathbb{F}_q)\}$.

The union is disjoint, except for the point $(0 : \ldots : 0 : 1) = L_0 \cap C_\infty$, thus S has $N := q^2 + 2(q + 1) - 1 = (q + 1)^2$ points hence C_d is a code of length $(q + 1)^2$. Since $\dim C_d = \dim_{\mathbb{F}_q} \mathbb{F}_q[\mathbf{X}]_d / I_S(d)$ to calculate $\dim C_d$ we could count the number of monomials of degree d in some footprint of I_S, so we look for a description of I_S.

A good place to start is to examine the ideal I generated by the set G of binomials "suggested" by the definition of S, namely
$G = \{X_i X_j - X_{i+1} X_{j-1} \mid 0 \leq i \leq \ell - 2\, , i \neq n, i + 1 < j \leq \ell\, , j \neq n + 1\}$

It turns out that G is a Gröbner basis for I w.r.t. the graded lexicographic order $\prec$ where $X_\ell \prec \cdots \prec X_0$, but then $I \subsetneq I_S$. Take for example $1 = m < n = 2$ and $q = 5$. We have $X_0 X_2 - X_1^2 \in G$ and if $I = I_S$ then we should have $X_0^5 X_2 - X_1^6 \in I$, but in this case $X_0^5 X_2 - X_1^6 - X_0^4(X_0 X_2 - X_1^2) = X_0^4 X_1^2 - X_1^6 \in I$ but this is not true (since $X_0^4 X_1^2$ is not a multiple of $X_0 X_2$, $X_0 X_4$ or $X_1 X_4$).

Codes defined on rational normal scrolls

It is not difficult to prove that $S = S_0 \cup C_\infty \cup L_0$, where
$S_0 = \{(1 : a : \cdots : a^n : b : ab : \cdots : a^m b) \mid (a, b) \in \mathbb{A}^2(\mathbb{F}_q)\}$,
$C_\infty = \{(0 : \cdots : 0 : a^m : a^{m-1}b : \cdots : b^m) \in S \mid (a : b) \in \mathbb{P}^1(\mathbb{F}_q)\}$, and
$L_0 = \{(0 : \cdots : 0 : a : 0 : \cdots 0 : b) \in S \mid (a : b) \in \mathbb{P}^1(\mathbb{F}_q)\}$.

The union is disjoint, except for the point $(0 : \ldots : 0 : 1) = L_0 \cap C_\infty$, thus S has $N := q^2 + 2(q + 1) - 1 = (q + 1)^2$ points hence C_d is a code of length $(q + 1)^2$. Since $\dim C_d = \dim_{\mathbb{F}_q} \mathbb{F}_q[\mathbf{X}]_d / I_S(d)$ to calculate $\dim C_d$ we could count the number of monomials of degree d in some footprint of I_S, so we look for a description of I_S.

A good place to start is to examine the ideal I generated by the set G of binomials "suggested" by the definition of S, namely
$G = \{X_i X_j - X_{i+1} X_{j-1} \mid 0 \leq i \leq \ell - 2\,,\, i \neq n, i + 1 < j \leq \ell\,,\, j \neq n + 1\}$

It turns out that G is a Gröbner basis for I w.r.t. the graded lexicographic order $\prec$ where $X_\ell \prec \cdots \prec X_0$, but then $I \subsetneq I_S$. Take for example $1 = m < n = 2$ and $q = 5$. We have $X_0X_2 - X_1^2 \in G$ and if $I = I_S$ then we should have $X_0^5 X_2 - X_1^6 \in I$, but in this case $X_0^5 X_2 - X_1^6 - X_0^4(X_0X_2 - X_1^2) = X_0^4 X_1^2 - X_1^6 \in I$ but this is not true (since $X_0^4 X_1^2$ is not a multiple of X_0X_2, X_0X_4 or X_1X_4).

Codes defined on rational normal scrolls

It is not difficult to prove that $S = S_0 \cup C_\infty \cup L_0$, where
$S_0 = \{(1 : a : \cdots : a^n : b : ab : \cdots : a^m b) \mid (a, b) \in \mathbb{A}^2(\mathbb{F}_q)\}$,
$C_\infty = \{(0 : \cdots : 0 : a^m : a^{m-1}b : \cdots : b^m) \in S \mid (a : b) \in \mathbb{P}^1(\mathbb{F}_q)\}$, and
$L_0 = \{(0 : \cdots : 0 : a : 0 : \cdots 0 : b) \in S \mid (a : b) \in \mathbb{P}^1(\mathbb{F}_q)\}$.

The union is disjoint, except for the point $(0 : \ldots : 0 : 1) = L_0 \cap C_\infty$, thus S has $N := q^2 + 2(q + 1) - 1 = (q + 1)^2$ points hence C_d is a code of length $(q + 1)^2$. Since $\dim C_d = \dim_{\mathbb{F}_q} \mathbb{F}_q[\mathbf{X}]_d / I_S(d)$ to calculate $\dim C_d$ we could count the number of monomials of degree d in some footprint of I_S, so we look for a description of I_S.

A good place to start is to examine the ideal I generated by the set G of binomials "suggested" by the definition of S, namely
$G = \{X_i X_j - X_{i+1} X_{j-1} \mid 0 \le i \le \ell - 2 \, , \, i \ne n, i + 1 < j \le \ell \, , \, j \ne n + 1\}$
It turns out that G is a Gröbner basis for I w.r.t. the graded lexicographic order $\prec$ where $X_\ell \prec \cdots \prec X_0$, but then $I \subsetneqq I_S$. Take for example $1 = m < n = 2$ and $q = 5$. We have $X_0 X_2 - X_1^2 \in G$ and if $I = I_S$ then we should have $X_0^5 X_2 - X_1^6 \in I$, but in this case $X_0^5 X_2 - X_1^6 - X_0^4(X_0 X_2 - X_1^2) = X_0^4 X_1^2 - X_1^6 \in I$ but this is not true (since $X_0^4 X_1^2$ is not a multiple of $X_0 X_2$, $X_0 X_4$ or $X_1 X_4$).

Codes defined on rational normal scrolls

It is not difficult to prove that $S = S_0 \cup C_\infty \cup L_0$, where
$S_0 = \{(1 : a : \cdots : a^n : b : ab : \cdots : a^m b) \mid (a,b) \in \mathbb{A}^2(\mathbb{F}_q)\}$,
$C_\infty = \{(0 : \cdots : 0 : a^m : a^{m-1}b : \cdots : b^m) \in S \mid (a : b) \in \mathbb{P}^1(\mathbb{F}_q)\}$, and
$L_0 = \{(0 : \cdots : 0 : a : 0 : \cdots 0 : b) \in S \mid (a : b) \in \mathbb{P}^1(\mathbb{F}_q)\}$.

The union is disjoint, except for the point $(0 : \ldots : 0 : 1) = L_0 \cap C_\infty$, thus S has $N := q^2 + 2(q+1) - 1 = (q+1)^2$ points hence C_d is a code of length $(q+1)^2$. Since $\dim C_d = \dim_{\mathbb{F}_q} \mathbb{F}_q[\mathbf{X}]_d / I_S(d)$ to calculate $\dim C_d$ we could count the number of monomials of degree d in some footprint of I_S, so we look for a description of I_S.

$$rank \begin{pmatrix} x_0 & x_1 & x_3 \\ x_1 & x_2 & x_4 \end{pmatrix} = 1$$

A good place to start is to examine the ideal I
binomials "suggested" by the definition of S, na

$G = \{X_i X_j - X_{i+1} X_{j-1} \mid 0 \le i \le \ell - 2\,,\, i \neq n, i+1 < j \le \ell\,,\, j \neq n+1\}$

It turns out that G is a Gröbner basis for I w.r.t. the graded lexicographic order $\prec$ where $X_\ell \prec \cdots \prec X_0$, but then $I \subsetneq I_S$. Take for example $1 = m < n = 2$ and $q = 5$. We have $X_0X_2 - X_1^2 \in G$ and if $I = I_S$ then we should have $X_0^5 X_2 - X_1^6 \in I$, but in this case $X_0^5 X_2 - X_1^6 - X_0^4(X_0X_2 - X_1^2) = X_0^4 X_1^2 - X_1^6 \in I$ but this is not true (since $X_0^4 X_1^2$ is not a multiple of X_0X_2, X_0X_4 or X_1X_4).

Codes defined on rational normal scrolls

It is not difficult to prove that $S = S_0 \cup C_\infty \cup L_0$, where
$S_0 = \{(1 : a : \cdots : a^n : b : ab : \cdots : a^m b) \mid (a, b) \in \mathbb{A}^2(\mathbb{F}_q)\}$,
$C_\infty = \{(0 : \cdots : 0 : a^m : a^{m-1}b : \cdots : b^m) \in S \mid (a : b) \in \mathbb{P}^1(\mathbb{F}_q)\}$, and
$L_0 = \{(0 : \cdots : 0 : a : 0 : \cdots 0 : b) \in S \mid (a : b) \in \mathbb{P}^1(\mathbb{F}_q)\}$.

The union is disjoint, except for the point $(0 : \ldots : 0 : 1) = L_0 \cap C_\infty$, thus S has $N := q^2 + 2(q+1) - 1 = (q+1)^2$ points hence C_d is a code of length $(q+1)^2$. Since $\dim C_d = \dim_{\mathbb{F}_q} \mathbb{F}_q[\mathbf{X}]_d / I_S(d)$ to calculate $\dim C_d$ we could count the number of monomials of degree d in some footprint of I_S, so we look for a description of I_S.

$$rank \begin{pmatrix} x_0 & x_1 & x_3 \\ x_1 & x_2 & x_4 \end{pmatrix} = 1$$

A good place to start is to examine the ideal I g
binomials "suggested" by the definition of S, na

$G = \{X_i X_j - X_{i+1} X_{j-1} \mid 0 \leq i \leq \ell - 2\, , i \neq n, i + 1 < j \leq \ell\, , j \neq n + 1\}$

It turns out that G is a Gröbner basis for I w.r.t. the graded lexicographic order $\prec$ where $X_\ell \prec \cdots \prec X_0$, but then $I \subsetneq I_S$. Take for example $1 = m < n = 2$ and $q = 5$. We have $X_0 X_2 - X_1^2 \in G$ and if $I = I_S$ then we should have $X_0^5 X_2 - X_1^6 \in I$, but in this case $X_0^5 X_2 - X_1^6 - X_0^4(X_0 X_2 - X_1^2) = X_0^4 X_1^2 - X_1^6 \in I$ but this is not true (since $X_0^4 X_1^2$ is not a multiple of $X_0 X_2$, $X_0 X_4$ or $X_1 X_4$).

Codes defined on rational normal scrolls

It is not difficult to prove that $S = S_0 \cup C_\infty \cup L_0$, where
$S_0 = \{(1 : a : \cdots : a^n : b : ab : \cdots : a^m b) \mid (a, b) \in \mathbb{A}^2(\mathbb{F}_q)\}$,
$C_\infty = \{(0 : \cdots : 0 : a^m : a^{m-1}b : \cdots : b^m) \in S \mid (a : b) \in \mathbb{P}^1(\mathbb{F}_q)\}$, and
$L_0 = \{(0 : \cdots : 0 : a : 0 : \cdots 0 : b) \in S \mid (a : b) \in \mathbb{P}^1(\mathbb{F}_q)\}$.

The union is disjoint, except for the point $(0 : \ldots : 0 : 1) = L_0 \cap C_\infty$, thus S has $N := q^2 + 2(q+1) - 1 = (q+1)^2$ points hence C_d is a code of length $(q+1)^2$. Since $\dim C_d = \dim_{\mathbb{F}_q} \mathbb{F}_q[\mathbf{X}]_d / I_S(d)$ to calculate $\dim C_d$ we could count the number of monomials of degree d in some footprint of I_S, so we look for a description of I_S.

$$rank \begin{pmatrix} x_0 & x_1 & x_3 \\ x_1 & x_2 & x_4 \end{pmatrix} = 1$$

A good place to start is to examine the ideal I g binomials "suggested" by the definition of S, na

$G = \{X_i X_j - X_{i+1} X_{j-1} \mid 0 \leq i \leq \ell - 2 \, , \, i \neq n, i + 1 < j \leq \ell \, , \, j \neq n + 1\}$

It turns out that G is a Gröbner basis for I w.r.t. the graded lexicographic order $\prec$ where $X_\ell \prec \cdots \prec X_0$, but then $I \subsetneq I_S$. Take for example $1 = m < n = 2$ and $q = 5$. We have $X_0 X_2 - X_1^2 \in G$ and if $I = I_S$ then we should have $X_0^5 X_2 - X_1^6 \in I$, but in this case
$X_0^5 X_2 - X_1^6 - X_0^4(X_0 X_2 - X_1^2) = X_0^4 X_1^2 - X_1^6 \in I$ but this is not true (since $X_0^4 X_1^2$ is not a multiple of $X_0 X_2$, $X_0 X_4$ or $X_1 X_4$).

Codes defined on rational normal scrolls

It is not difficult to prove that $S = S_0 \cup C_\infty \cup L_0$, where
$S_0 = \{(1 : a : \cdots : a^n : b : ab : \cdots : a^m b) \mid (a, b) \in \mathbb{A}^2(\mathbb{F}_q)\}$,
$C_\infty = \{(0 : \cdots : 0 : a^m : a^{m-1}b : \cdots : b^m) \in S \mid (a : b) \in \mathbb{P}^1(\mathbb{F}_q)\}$, and
$L_0 = \{(0 : \cdots : 0 : a : 0 : \cdots 0 : b) \in S \mid (a : b) \in \mathbb{P}^1(\mathbb{F}_q)\}$.

The union is disjoint, except for the point $(0 : \ldots : 0 : 1) = L_0 \cap C_\infty$, thus S has $N := q^2 + 2(q + 1) - 1 = (q + 1)^2$ points hence C_d is a code of length $(q + 1)^2$. Since $\dim C_d = \dim_{\mathbb{F}_q} \mathbb{F}_q[\mathbf{X}]_d / I_S(d)$ to calculate $\dim C_d$ we could count the number of monomials of degree d in some footprint of I_S, so we look for a description of I_S.

$$rank \begin{pmatrix} x_0 & x_1 & x_3 \\ x_1 & x_2 & x_4 \end{pmatrix} = 1$$

A good place to start is to examine the ideal I g
binomials "suggested" by the definition of S, na

$$G = \{X_i X_j - X_{i+1} X_{j-1} \mid 0 \le i \le \ell - 2\,,\, i \ne n, i + 1 < j \le \ell\,,\, j \ne n + 1\}$$

It turns out that G is a Gröbner basis for I w.r.t. the graded lexicographic order $\prec$ where $X_\ell \prec \cdots \prec X_0$, but then $I \subsetneq I_S$. Take for example $1 = m < n = 2$ and $q = 5$. We have $X_0 X_2 - X_1^2 \in G$ and if $I = I_S$ then we should have $X_0^5 X_2 - X_1^6 \in I$, but in this case $X_0^5 X_2 - X_1^6 - X_0^4(X_0 X_2 - X_1^2) = X_0^4 X_1^2 - X_1^6 \in I$ but this is not true (since $X_0^4 X_1^2$ is not a multiple of $X_0 X_2$, $X_0 X_4$ or $X_1 X_4$).

Codes defined on rational normal scrolls

We found a parametrization for S using four parameters $\alpha, \beta, \gamma, \delta \in \mathbb{F}_q$ in the following way:

$$rank \begin{pmatrix} a_0 & a_1 & \cdots & a_{n-1} & a_{n+1} & \cdots & a_{\ell-1} \\ a_1 & a_2 & \cdots & a_n & a_{n+2} & \cdots & a_\ell \end{pmatrix} = 1$$

This lead us to consider the homomorphism of $\mathbb{F}_q$-algebras $\Psi : \mathbb{F}_q[\mathbf{X}] \longrightarrow \mathbb{F}_q[Y, Z, V, W]$ given by $\Psi(X_i) = YV^iW^{n-i}$ for $i = 0, \ldots, n$ and $\Psi(X_j) = ZV^{j-n-1}W^{\ell-j}$ for $j = n+1, \ldots, \ell = n+m+1$.

Let $\mathcal{B}$ be the image of Ψ and let's consider $\Psi : \mathbb{F}_q[\mathbf{X}] \longrightarrow \mathcal{B}$.

We can make Ψ a graded morphism of algebras by defining $\deg(Y^yZ^zV^vW^w) = y + z$, and then $\mathcal{B} = \bigoplus_{d \geq 0} \mathcal{B}_d$, where $\mathcal{B}_d$ is generated as an $\mathbb{F}_q$-module by $Y^yZ^zV^vW^w$ with $y + z = d$ and $v + w = ny + mz$.

Codes defined on rational normal scrolls

We found a parametrization for S using four parameters $\alpha, \beta, \gamma, \delta \in \mathbb{F}_q$ in the following way:

$$\text{rank} \begin{pmatrix} a_0 & a_1 & \cdots & a_{n-1} & a_{n+1} & \cdots & a_{\ell-1} \\ a_1 & a_2 & \cdots & a_n & a_{n+2} & \cdots & a_\ell \end{pmatrix} = 1$$

This lead us to consider the homomorphism of $\mathbb{F}_q$-algebras $\Psi : \mathbb{F}_q[\mathbf{X}] \longrightarrow \mathbb{F}_q[Y, Z, V, W]$ given by $\Psi(X_i) = YV^i W^{n-i}$ for $i = 0, \ldots, n$ and $\Psi(X_j) = ZV^{j-n-1} W^{\ell-j}$ for $j = n+1, \ldots, \ell = n+m+1$.

Let $\mathcal{B}$ be the image of Ψ and let's consider $\Psi : \mathbb{F}_q[\mathbf{X}] \longrightarrow \mathcal{B}$.

We can make Ψ a graded morphism of algebras by defining $\deg(Y^y Z^z V^v W^w) = y + z$, and then $\mathcal{B} = \bigoplus_{d \geq 0} \mathcal{B}_d$, where $\mathcal{B}_d$ is generated as an $\mathbb{F}_q$-module by $Y^y Z^z V^v W^w$ with $y + z = d$ and $v + w = ny + mz$.

Codes defined on rational normal scrolls

We found a parametrization for S using four parameters $\alpha, \beta, \gamma, \delta \in \mathbb{F}_q$ in the following way:

$$(\; a_0 \; : \cdots : \; a_n \; : \; a_{n+1} \; : \cdots : \; a_\ell \;) \in S \text{ if and only if}$$

$$rank \begin{pmatrix} a_0 & a_1 & \cdots & a_{n-1} & a_{n+1} & \cdots & a_{\ell-1} \\ a_1 & a_2 & \cdots & a_n & a_{n+2} & \cdots & a_\ell \end{pmatrix} = 1$$

This lead us to consider the homomorphism of $\mathbb{F}_q$-algebras $\Psi : \mathbb{F}_q[\mathbf{X}] \longrightarrow \mathbb{F}_q[Y, Z, V, W]$ given by $\Psi(X_i) = YV^iW^{n-i}$ for $i = 0, \ldots, n$ and $\Psi(X_j) = ZV^{j-n-1}W^{\ell-j}$ for $j = n+1, \ldots, \ell = n+m+1$.

Let $\mathcal{B}$ be the image of Ψ and let's consider $\Psi : \mathbb{F}_q[\mathbf{X}] \longrightarrow \mathcal{B}$.

We can make Ψ a graded morphism of algebras by defining $\deg(Y^yZ^zV^vW^w) = y + z$, and then $\mathcal{B} = \bigoplus_{d \geq 0} \mathcal{B}_d$, where $\mathcal{B}_d$ is generated as an $\mathbb{F}_q$-module by $Y^yZ^zV^vW^w$ with $y + z = d$ and $v + w = ny + mz$.

Codes defined on rational normal scrolls

We found a parametrization for S using four parameters $\alpha, \beta, \gamma, \delta \in \mathbb{F}_q$ in the following way:

$$(\ \alpha\delta^n : \cdots : a_n : a_{n+1} : \cdots : a_\ell\ ) \in S \text{ if and only if}$$

$$rank \begin{pmatrix} \alpha\delta^n & a_1 & \cdots & a_{n-1} & a_{n+1} & \cdots & a_{\ell-1} \\ a_1 & a_2 & \cdots & a_n & a_{n+2} & \cdots & a_\ell \end{pmatrix} = 1$$

This lead us to consider the homomorphism of $\mathbb{F}_q$-algebras $\Psi : \mathbb{F}_q[\mathbf{X}] \longrightarrow \mathbb{F}_q[Y, Z, V, W]$ given by $\Psi(X_i) = YV^iW^{n-i}$ for $i = 0, \ldots, n$ and $\Psi(X_j) = ZV^{j-n-1}W^{\ell-j}$ for $j = n+1, \ldots, \ell = n+m+1$.

Let $\mathcal{B}$ be the image of Ψ and let's consider $\Psi : \mathbb{F}_q[\mathbf{X}] \longrightarrow \mathcal{B}$.

We can make Ψ a graded morphism of algebras by defining $\deg(Y^yZ^zV^vW^w) = y + z$, and then $\mathcal{B} = \bigoplus_{d \geq 0} \mathcal{B}_d$, where $\mathcal{B}_d$ is generated as an $\mathbb{F}_q$-module by $Y^yZ^zV^vW^w$ with $y + z = d$ and $v + w = ny + mz$.

Codes defined on rational normal scrolls

We found a parametrization for S using four parameters $\alpha, \beta, \gamma, \delta \in \mathbb{F}_q$ in the following way:

$$(\ \alpha\delta^n \ : \cdots : \ a_n \ : \ a_{n+1} \ : \cdots : \ a_\ell \ ) \in S \text{ if and only if}$$

$$rank \begin{pmatrix} \alpha\delta^n & \alpha\gamma\delta^{n-1} & \cdots & a_{n-1} & a_{n+1} & \cdots & a_{\ell-1} \\ \alpha\gamma\delta^{n-1} & a_2 & \cdots & a_n & a_{n+2} & \cdots & a_\ell \end{pmatrix} = 1$$

This lead us to consider the homomorphism of $\mathbb{F}_q$-algebras $\Psi : \mathbb{F}_q[\mathbf{X}] \longrightarrow \mathbb{F}_q[Y, Z, V, W]$ given by $\Psi(X_i) = YV^iW^{n-i}$ for $i = 0, \ldots, n$ and $\Psi(X_j) = ZV^{j-n-1}W^{\ell-j}$ for $j = n+1, \ldots, \ell = n+m+1$.

Let $\mathcal{B}$ be the image of Ψ and let's consider $\Psi : \mathbb{F}_q[\mathbf{X}] \longrightarrow \mathcal{B}$.

We can make Ψ a graded morphism of algebras by defining $\deg(Y^yZ^zV^vW^w) = y + z$, and then $\mathcal{B} = \bigoplus_{d \geq 0} \mathcal{B}_d$, where $\mathcal{B}_d$ is generated as an $\mathbb{F}_q$-module by $Y^yZ^zV^vW^w$ with $y + z = d$ and $v + w = ny + mz$.

Codes defined on rational normal scrolls

We found a parametrization for S using four parameters $\alpha, \beta, \gamma, \delta \in \mathbb{F}_q$ in the following way:

$$(\ \alpha\delta^n \ : \cdots : \ a_n \ : \ a_{n+1} \ : \cdots : \ a_\ell \ ) \in S \text{ if and only if}$$

$$rank \begin{pmatrix} \alpha\delta^n & \alpha\gamma\delta^{n-1} & \cdots & a_{n-1} & a_{n+1} & \cdots & a_{\ell-1} \\ \alpha\gamma\delta^{n-1} & \alpha\gamma^2\delta^{n-2} & \cdots & a_n & a_{n+2} & \cdots & a_\ell \end{pmatrix} = 1$$

This lead us to consider the homomorphism of $\mathbb{F}_q$-algebras $\Psi : \mathbb{F}_q[\mathbf{X}] \longrightarrow \mathbb{F}_q[Y, Z, V, W]$ given by $\Psi(X_i) = YV^iW^{n-i}$ for $i = 0, \ldots, n$ and $\Psi(X_j) = ZV^{j-n-1}W^{\ell-j}$ for $j = n+1, \ldots, \ell = n+m+1$.

Let $\mathcal{B}$ be the image of Ψ and let's consider $\Psi : \mathbb{F}_q[\mathbf{X}] \longrightarrow \mathcal{B}$.

We can make Ψ a graded morphism of algebras by defining $\deg(Y^yZ^zV^vW^w) = y + z$, and then $\mathcal{B} = \bigoplus_{d \geq 0} \mathcal{B}_d$, where $\mathcal{B}_d$ is generated as an $\mathbb{F}_q$-module by $Y^yZ^zV^vW^w$ with $y + z = d$ and $v + w = ny + mz$.

Codes defined on rational normal scrolls

We found a parametrization for S using four parameters $\alpha, \beta, \gamma, \delta \in \mathbb{F}_q$ in the following way:

$$(\ \alpha\delta^n \ : \cdots : \ a_n \ : \ a_{n+1} \ : \cdots : \ a_\ell \ ) \in S \text{ if and only if}$$

$$rank \begin{pmatrix} \alpha\delta^n & \alpha\gamma\delta^{n-1} & \cdots & \alpha\gamma^{n-1}\delta & a_{n+1} & \cdots & a_{\ell-1} \\ \alpha\gamma\delta^{n-1} & \alpha\gamma^2\delta^{n-2} & \cdots & a_n & a_{n+2} & \cdots & a_\ell \end{pmatrix} = 1$$

This lead us to consider the homomorphism of $\mathbb{F}_q$-algebras $\Psi : \mathbb{F}_q[\mathbf{X}] \longrightarrow \mathbb{F}_q[Y, Z, V, W]$ given by $\Psi(X_i) = YV^iW^{n-i}$ for $i = 0, \ldots, n$ and $\Psi(X_j) = ZV^{j-n-1}W^{\ell-j}$ for $j = n+1, \ldots, \ell = n+m+1$.

Let $\mathcal{B}$ be the image of Ψ and let's consider $\Psi : \mathbb{F}_q[\mathbf{X}] \longrightarrow \mathcal{B}$.

We can make Ψ a graded morphism of algebras by defining $\deg(Y^yZ^zV^vW^w) = y + z$, and then $\mathcal{B} = \bigoplus_{d \geq 0} \mathcal{B}_d$, where $\mathcal{B}_d$ is generated as an $\mathbb{F}_q$-module by $Y^yZ^zV^vW^w$ with $y + z = d$ and $v + w = ny + mz$.

Codes defined on rational normal scrolls

We found a parametrization for S using four parameters $\alpha, \beta, \gamma, \delta \in \mathbb{F}_q$ in the following way:

$$(\alpha\delta^n : \cdots : \alpha\gamma^n : a_{n+1} : \cdots : a_\ell) \in S \text{ if and only if}$$

$$rank \begin{pmatrix} \alpha\delta^n & \alpha\gamma\delta^{n-1} & \cdots & \alpha\gamma^{n-1}\delta & a_{n+1} & \cdots & a_{\ell-1} \\ \alpha\gamma\delta^{n-1} & \alpha\gamma^2\delta^{n-2} & \cdots & \alpha\gamma^n & a_{n+2} & \cdots & a_\ell \end{pmatrix} = 1$$

This lead us to consider the homomorphism of $\mathbb{F}_q$-algebras $\Psi : \mathbb{F}_q[\mathbf{X}] \longrightarrow \mathbb{F}_q[Y, Z, V, W]$ given by $\Psi(X_i) = YV^iW^{n-i}$ for $i = 0, \ldots, n$ and $\Psi(X_j) = ZV^{j-n-1}W^{\ell-j}$ for $j = n+1, \ldots, \ell = n+m+1$.

Let $\mathcal{B}$ be the image of Ψ and let's consider $\Psi : \mathbb{F}_q[\mathbf{X}] \longrightarrow \mathcal{B}$.

We can make Ψ a graded morphism of algebras by defining $\deg(Y^yZ^zV^vW^w) = y + z$, and then $\mathcal{B} = \bigoplus_{d \geq 0} \mathcal{B}_d$, where $\mathcal{B}_d$ is generated as an $\mathbb{F}_q$-module by $Y^yZ^zV^vW^w$ with $y + z = d$ and $v + w = ny + mz$.

Codes defined on rational normal scrolls

We found a parametrization for S using four parameters $\alpha, \beta, \gamma, \delta \in \mathbb{F}_q$ in the following way:

$$(\ \alpha\delta^n \ : \cdots : \ \alpha\gamma^n \ : \ \beta\delta^m \ : \cdots : \ a_\ell \ ) \in S \text{ if and only if}$$

$$rank \begin{pmatrix} \alpha\delta^n & \alpha\gamma\delta^{n-1} & \cdots & \alpha\gamma^{n-1}\delta & \beta\delta^m & \cdots & a_{\ell-1} \\ \alpha\gamma\delta^{n-1} & \alpha\gamma^2\delta^{n-2} & \cdots & \alpha\gamma^n & a_{n+2} & \cdots & a_\ell \end{pmatrix} = 1$$

This lead us to consider the homomorphism of $\mathbb{F}_q$-algebras $\Psi : \mathbb{F}_q[\mathbf{X}] \longrightarrow \mathbb{F}_q[Y, Z, V, W]$ given by $\Psi(X_i) = YV^iW^{n-i}$ for $i = 0, \ldots, n$ and $\Psi(X_j) = ZV^{j-n-1}W^{\ell-j}$ for $j = n+1, \ldots, \ell = n+m+1$.

Let $\mathcal{B}$ be the image of Ψ and let's consider $\Psi : \mathbb{F}_q[\mathbf{X}] \longrightarrow \mathcal{B}$.

We can make Ψ a graded morphism of algebras by defining $\deg(Y^yZ^zV^vW^w) = y + z$, and then $\mathcal{B} = \bigoplus_{d \geq 0} \mathcal{B}_d$, where $\mathcal{B}_d$ is generated as an $\mathbb{F}_q$-module by $Y^yZ^zV^vW^w$ with $y + z = d$ and $v + w = ny + mz$.

Codes defined on rational normal scrolls

We found a parametrization for S using four parameters $\alpha, \beta, \gamma, \delta \in \mathbb{F}_q$ in the following way:

$$(\ \alpha\delta^n \ : \cdots : \ \alpha\gamma^n \ : \ \beta\delta^m \ : \cdots : \ a_\ell \ ) \in S \text{ if and only if}$$

$$rank \begin{pmatrix} \alpha\delta^n & \alpha\gamma\delta^{n-1} & \cdots & \alpha\gamma^{n-1}\delta & \beta\delta^m & \cdots & a_{\ell-1} \\ \alpha\gamma\delta^{n-1} & \alpha\gamma^2\delta^{n-2} & \cdots & \alpha\gamma^n & \beta\gamma\delta^{m-1} & \cdots & a_\ell \end{pmatrix} = 1$$

This lead us to consider the homomorphism of $\mathbb{F}_q$-algebras $\Psi : \mathbb{F}_q[\mathbf{X}] \longrightarrow \mathbb{F}_q[Y, Z, V, W]$ given by $\Psi(X_i) = YV^iW^{n-i}$ for $i = 0, \ldots, n$ and $\Psi(X_j) = ZV^{j-n-1}W^{\ell-j}$ for $j = n+1, \ldots, \ell = n+m+1$.

Let $\mathcal{B}$ be the image of Ψ and let's consider $\Psi : \mathbb{F}_q[\mathbf{X}] \longrightarrow \mathcal{B}$.

We can make Ψ a graded morphism of algebras by defining $\deg(Y^yZ^zV^vW^w) = y + z$, and then $\mathcal{B} = \bigoplus_{d \geq 0} \mathcal{B}_d$, where $\mathcal{B}_d$ is generated as an $\mathbb{F}_q$-module by $Y^yZ^zV^vW^w$ with $y + z = d$ and $v + w = ny + mz$.

Codes defined on rational normal scrolls

We found a parametrization for S using four parameters $\alpha, \beta, \gamma, \delta \in \mathbb{F}_q$ in the following way:

$$(\alpha\delta^n : \cdots : \alpha\gamma^n : \beta\delta^m : \cdots : a_\ell) \in S \text{ if and only if}$$

$$rank \begin{pmatrix} \alpha\delta^n & \alpha\gamma\delta^{n-1} & \cdots & \alpha\gamma^{n-1}\delta & \beta\delta^m & \cdots & \beta\gamma^{m-1}\delta \\ \alpha\gamma\delta^{n-1} & \alpha\gamma^2\delta^{n-2} & \cdots & \alpha\gamma^n & \beta\gamma\delta^{m-1} & \cdots & a_\ell \end{pmatrix} = 1$$

This lead us to consider the homomorphism of $\mathbb{F}_q$-algebras $\Psi : \mathbb{F}_q[\mathbf{X}] \longrightarrow \mathbb{F}_q[Y, Z, V, W]$ given by $\Psi(X_i) = YV^iW^{n-i}$ for $i = 0, \ldots, n$ and $\Psi(X_j) = ZV^{j-n-1}W^{\ell-j}$ for $j = n+1, \ldots, \ell = n+m+1$.

Let $\mathcal{B}$ be the image of Ψ and let's consider $\Psi : \mathbb{F}_q[\mathbf{X}] \longrightarrow \mathcal{B}$.

We can make Ψ a graded morphism of algebras by defining $\deg(Y^yZ^zV^vW^w) = y + z$, and then $\mathcal{B} = \bigoplus_{d \geq 0} \mathcal{B}_d$, where $\mathcal{B}_d$ is generated as an $\mathbb{F}_q$-module by $Y^yZ^zV^vW^w$ with $y + z = d$ and $v + w = ny + mz$.

Codes defined on rational normal scrolls

We found a parametrization for S using four parameters $\alpha, \beta, \gamma, \delta \in \mathbb{F}_q$ in the following way:

$$(\alpha\delta^n : \cdots : \alpha\gamma^n : \beta\delta^m : \cdots : \beta\gamma^m) \in S \text{ if and only if}$$

$$rank \begin{pmatrix} \alpha\delta^n & \alpha\gamma\delta^{n-1} & \cdots & \alpha\gamma^{n-1}\delta & \beta\delta^m & \cdots & \beta\gamma^{m-1}\delta \\ \alpha\gamma\delta^{n-1} & \alpha\gamma^2\delta^{n-2} & \cdots & \alpha\gamma^n & \beta\gamma\delta^{m-1} & \cdots & \beta\gamma^m \end{pmatrix} = 1$$

This lead us to consider the homomorphism of $\mathbb{F}_q$-algebras $\Psi : \mathbb{F}_q[\mathbf{X}] \longrightarrow \mathbb{F}_q[Y, Z, V, W]$ given by $\Psi(X_i) = YV^iW^{n-i}$ for $i = 0, \ldots, n$ and $\Psi(X_j) = ZV^{j-n-1}W^{\ell-j}$ for $j = n+1, \ldots, \ell = n+m+1$.

Let $\mathcal{B}$ be the image of Ψ and let's consider $\Psi : \mathbb{F}_q[\mathbf{X}] \longrightarrow \mathcal{B}$.

We can make Ψ a graded morphism of algebras by defining $\deg(Y^yZ^zV^vW^w) = y + z$, and then $\mathcal{B} = \bigoplus_{d \geq 0} \mathcal{B}_d$, where $\mathcal{B}_d$ is generated as an $\mathbb{F}_q$-module by $Y^yZ^zV^vW^w$ with $y + z = d$ and $v + w = ny + mz$.

Codes defined on rational normal scrolls

We found a parametrization for S using four parameters $\alpha, \beta, \gamma, \delta \in \mathbb{F}_q$ in the following way:

$$(\; \alpha\delta^n \; : \cdots : \; \alpha\gamma^n \; : \; \beta\delta^m \; : \cdots : \; \beta\gamma^m \;) \in S \text{ if and only if}$$

$$rank \begin{pmatrix} \alpha\delta^n & \alpha\gamma\delta^{n-1} & \cdots & \alpha\gamma^{n-1}\delta & \beta\delta^m & \cdots & \beta\gamma^{m-1}\delta \\ \alpha\gamma\delta^{n-1} & \alpha\gamma^2\delta^{n-2} & \cdots & \alpha\gamma^n & \beta\gamma\delta^{m-1} & \cdots & \beta\gamma^m \end{pmatrix} = 1$$

This lead us to consider the homomorphism of $\mathbb{F}_q$-algebras $\Psi : \mathbb{F}_q[\mathbf{X}] \longrightarrow \mathbb{F}_q[Y, Z, V, W]$ given by $\Psi(X_i) = YV^iW^{n-i}$ for $i = 0, \ldots, n$ and $\Psi(X_j) = ZV^{j-n-1}W^{\ell-j}$ for $j = n+1, \ldots, \ell = n+m+1$.

Let $\mathcal{B}$ be the image of Ψ and let's consider $\Psi : \mathbb{F}_q[\mathbf{X}] \longrightarrow \mathcal{B}$.

We can make Ψ a graded morphism of algebras by defining $\deg(Y^yZ^zV^vW^w) = y + z$, and then $\mathcal{B} = \bigoplus_{d \geq 0} \mathcal{B}_d$, where $\mathcal{B}_d$ is generated as an $\mathbb{F}_q$-module by $Y^yZ^zV^vW^w$ with $y + z = d$ and $v + w = ny + mz$.

Codes defined on rational normal scrolls

We found a parametrization for S using four parameters $\alpha, \beta, \gamma, \delta \in \mathbb{F}_q$ in the following way:

$$(\ \alpha\delta^n \ : \cdots : \ \alpha\gamma^n \ : \ \beta\delta^m \ : \cdots : \ \beta\gamma^m \ ) \in S \text{ if and only if}$$

$$rank \begin{pmatrix} \alpha\delta^n & \alpha\gamma\delta^{n-1} & \cdots & \alpha\gamma^{n-1}\delta & \beta\delta^m & \cdots & \beta\gamma^{m-1}\delta \\ \alpha\gamma\delta^{n-1} & \alpha\gamma^2\delta^{n-2} & \cdots & \alpha\gamma^n & \beta\gamma\delta^{m-1} & \cdots & \beta\gamma^m \end{pmatrix} = 1$$

This lead us to consider the homomorphism of $\mathbb{F}_q$-algebras $\Psi : \mathbb{F}_q[\mathbf{X}] \longrightarrow \mathbb{F}_q[Y, Z, V, W]$ given by $\Psi(X_i) = YV^iW^{n-i}$ for $i = 0, \ldots, n$ and $\Psi(X_j) = ZV^{j-n-1}W^{\ell-j}$ for $j = n+1, \ldots, \ell = n+m+1$.

Let $\mathcal{B}$ be the image of Ψ and let's consider $\Psi : \mathbb{F}_q[\mathbf{X}] \longrightarrow \mathcal{B}$.

We can make Ψ a graded morphism of algebras by defining $\deg(Y^yZ^zV^vW^w) = y + z$, and then $\mathcal{B} = \bigoplus_{d \geq 0} \mathcal{B}_d$, where $\mathcal{B}_d$ is generated as an $\mathbb{F}_q$-module by $Y^yZ^zV^vW^w$ with $y + z = d$ and $v + w = ny + mz$.

Codes defined on rational normal scrolls

We found a parametrization for S using four parameters $\alpha, \beta, \gamma, \delta \in \mathbb{F}_q$ in the following way:

$$(\ \alpha\delta^n \ : \cdots : \ \alpha\gamma^n \ : \ \beta\delta^m \ : \cdots : \ \beta\gamma^m \ ) \in S \text{ if and only if}$$

$$rank \begin{pmatrix} \alpha\delta^n & \alpha\gamma\delta^{n-1} & \cdots & \alpha\gamma^{n-1}\delta & \beta\delta^m & \cdots & \beta\gamma^{m-1}\delta \\ \alpha\gamma\delta^{n-1} & \alpha\gamma^2\delta^{n-2} & \cdots & \alpha\gamma^n & \beta\gamma\delta^{m-1} & \cdots & \beta\gamma^m \end{pmatrix} = 1$$

This lead us to consider the homomorphism of $\mathbb{F}_q$-algebras $\Psi : \mathbb{F}_q[\mathbf{X}] \longrightarrow \mathbb{F}_q[Y, Z, V, W]$ given by $\Psi(X_i) = YV^iW^{n-i}$ for $i = 0, \ldots, n$ and $\Psi(X_j) = ZV^{j-n-1}W^{\ell-j}$ for $j = n+1, \ldots, \ell = n+m+1$.

Let $\mathcal{B}$ be the image of Ψ and let's consider $\Psi : \mathbb{F}_q[\mathbf{X}] \longrightarrow \mathcal{B}$.

We can make Ψ a graded morphism of algebras by defining $\deg(Y^yZ^zV^vW^w) = y + z$, and then $\mathcal{B} = \bigoplus_{d \geq 0} \mathcal{B}_d$, where $\mathcal{B}_d$ is generated as an $\mathbb{F}_q$-module by $Y^yZ^zV^vW^w$ with $y + z = d$ and $v + w = ny + mz$.

Codes defined on rational normal scrolls

We found a parametrization for S using four parameters $\alpha, \beta, \gamma, \delta \in \mathbb{F}_q$ in the following way:

$$
\begin{array}{cccccccc}
 & YV^0W^n & YV^1W^{n-1} & & YV^{n-1}W^1 & ZV^0W^m & & ZV^{m-1}W^1 \\
rank & \alpha\delta^n & \alpha\gamma\delta^{n-1} & \cdots & \alpha\gamma^{n-1}\delta & \beta\delta^m & \cdots & \beta\gamma^{m-1}\delta \\
 & \alpha\gamma\delta^{n-1} & \alpha\gamma^2\delta^{n-2} & \cdots & \alpha\gamma^n & \beta\gamma\delta^{m-1} & \cdots & \beta\gamma^m \\
 & YV^1W^{n-1} & YV^2W^{n-2} & & YV^0W^n & ZV^1W^{m-1} & & ZV^mW^0
\end{array} = 1
$$

This lead us to consider the homomorphism of $\mathbb{F}_q$-algebras $\Psi : \mathbb{F}_q[\mathbf{X}] \longrightarrow \mathbb{F}_q[Y, Z, V, W]$ given by $\Psi(X_i) = YV^iW^{n-i}$ for $i = 0, \ldots, n$ and $\Psi(X_j) = ZV^{j-n-1}W^{\ell-j}$ for $j = n+1, \ldots, \ell = n+m+1$.

Let $\mathcal{B}$ be the image of Ψ and let's consider $\Psi : \mathbb{F}_q[\mathbf{X}] \longrightarrow \mathcal{B}$.

We can make Ψ a graded morphism of algebras by defining $\deg(Y^yZ^zV^vW^w) = y + z$, and then $\mathcal{B} = \bigoplus_{d \geq 0} \mathcal{B}_d$, where $\mathcal{B}_d$ is generated as an $\mathbb{F}_q$-module by $Y^yZ^zV^vW^w$ with $y + z = d$ and $v + w = ny + mz$.

Codes defined on rational normal scrolls

We found a parametrization for S using four parameters $\alpha, \beta, \gamma, \delta \in \mathbb{F}_q$ in the following way:

$$rank \begin{pmatrix} \overset{YV^0W^n}{\alpha\delta^n} & \overset{YV^1W^{n-1}}{\alpha\gamma\delta^{n-1}} & \cdots & \overset{YV^{n-1}W^1}{\alpha\gamma^{n-1}\delta} & \overset{ZV^0W^m}{\beta\delta^m} & \cdots & \overset{ZV^{m-1}W^1}{\beta\gamma^{m-1}\delta} \\ \underset{YV^1W^{n-1}}{\alpha\gamma\delta^{n-1}} & \underset{YV^2W^{n-2}}{\alpha\gamma^2\delta^{n-2}} & \cdots & \underset{YV^0W^n}{\alpha\gamma^n} & \underset{ZV^1W^{m-1}}{\beta\gamma\delta^{m-1}} & \cdots & \underset{ZV^mW^0}{\beta\gamma^m} \end{pmatrix} = 1$$

This lead us to consider the homomorphism of $\mathbb{F}_q$-algebras $\Psi : \mathbb{F}_q[\mathbf{X}] \longrightarrow \mathbb{F}_q[Y, Z, V, W]$ given by $\Psi(X_i) = YV^iW^{n-i}$ for $i = 0, \ldots, n$ and $\Psi(X_j) = ZV^{j-n-1}W^{\ell-j}$ for $j = n+1, \ldots, \ell = n+m+1$.

Let $\mathcal{B}$ be the image of Ψ and let's consider $\Psi : \mathbb{F}_q[\mathbf{X}] \longrightarrow \mathcal{B}$.

We can make Ψ a graded morphism of algebras by defining $\deg(Y^yZ^zV^vW^w) = y + z$, and then $\mathcal{B} = \bigoplus_{d \geq 0} \mathcal{B}_d$, where $\mathcal{B}_d$ is generated as an $\mathbb{F}_q$-module by $Y^yZ^zV^vW^w$ with $y + z = d$ and $v + w = ny + mz$.

Codes defined on rational normal scrolls

We found a parametrization for S using four parameters $\alpha, \beta, \gamma, \delta \in \mathbb{F}_q$ in the following way:

$$\begin{matrix} & YV^0W^n & YV^1W^{n-1} & & YV^{n-1}W^1 & ZV^0W^m & & ZV^{m-1}W^1 \\ rank & \alpha\delta^n & \alpha\gamma\delta^{n-1} & \cdots & \alpha\gamma^{n-1}\delta & \beta\delta^m & \cdots & \beta\gamma^{m-1}\delta \\ & \alpha\gamma\delta^{n-1} & \alpha\gamma^2\delta^{n-2} & \cdots & \alpha\gamma^n & \beta\gamma\delta^{m-1} & \cdots & \beta\gamma^m \\ & YV^1W^{n-1} & YV^2W^{n-2} & & YV^0W^n & ZV^1W^{m-1} & & ZV^mW^0 \end{matrix} = 1$$

This lead us to consider the homomorphism of $\mathbb{F}_q$-algebras $\Psi : \mathbb{F}_q[\mathbf{X}] \longrightarrow \mathbb{F}_q[Y, Z, V, W]$ given by $\Psi(X_i) = YV^iW^{n-i}$ for $i = 0, \ldots, n$ and $\Psi(X_j) = ZV^{j-n-1}W^{\ell-j}$ for $j = n+1, \ldots, \ell = n+m+1$.

Let $\mathcal{B}$ be the image of Ψ and let's consider $\Psi : \mathbb{F}_q[\mathbf{X}] \longrightarrow \mathcal{B}$.

We can make Ψ a graded morphism of algebras by defining $\deg(Y^yZ^zV^vW^w) = y + z$, and then $\mathcal{B} = \bigoplus_{d\geq 0} \mathcal{B}_d$, where $\mathcal{B}_d$ is generated as an $\mathbb{F}_q$-module by $Y^yZ^zV^vW^w$ with $y + z = d$ and $v + w = ny + mz$.

Codes defined on rational normal scrolls

We found a parametrization for S using four parameters $\alpha, \beta, \gamma, \delta \in \mathbb{F}_q$ in the following way:

$$\begin{matrix} & YV^0W^n & YV^1W^{n-1} & & YV^{n-1}W^1 & ZV^0W^m & & ZV^{m-1}W^1 \\ rank & \alpha\delta^n & \alpha\gamma\delta^{n-1} & \cdots & \alpha\gamma^{n-1}\delta & \beta\delta^m & \cdots & \beta\gamma^{m-1}\delta \\ & \alpha\gamma\delta^{n-1} & \alpha\gamma^2\delta^{n-2} & \cdots & \alpha\gamma^n & \beta\gamma\delta^{m-1} & \cdots & \beta\gamma^m \\ & YV^1W^{n-1} & YV^2W^{n-2} & & YV^0W^n & ZV^1W^{m-1} & & ZV^mW^0 \end{matrix} = 1$$

This lead us to consider the homomorphism of $\mathbb{F}_q$-algebras $\Psi : \mathbb{F}_q[\mathbf{X}] \longrightarrow \mathbb{F}_q[Y, Z, V, W]$ given by $\Psi(X_i) = YV^iW^{n-i}$ for $i = 0, \ldots, n$ and $\Psi(X_j) = ZV^{j-n-1}W^{\ell-j}$ for $j = n+1, \ldots, \ell = n+m+1$.

Let $\mathcal{B}$ be the image of Ψ and let's consider $\Psi : \mathbb{F}_q[\mathbf{X}] \longrightarrow \mathcal{B}$.

We can make Ψ a graded morphism of algebras by defining $\deg(Y^yZ^zV^vW^w) = y + z$, and then $\mathcal{B} = \bigoplus_{d \geq 0} \mathcal{B}_d$, where $\mathcal{B}_d$ is generated as an $\mathbb{F}_q$-module by $Y^yZ^zV^vW^w$ with $y + z = d$ and $v + w = ny + mz$.

Codes defined on rational normal scrolls

We found a parametrization for S using four parameters $\alpha, \beta, \gamma, \delta \in \mathbb{F}_q$ in the following way:

$$\begin{array}{cccccccc} & YV^0W^n & YV^1W^{n-1} & & YV^{n-1}W^1 & ZV^0W^m & & ZV^{m-1}W^1 \\ rank & \Bigg(\begin{array}{c} \alpha\delta^n \\ \alpha\gamma\delta^{n-1} \end{array} & \begin{array}{c} \alpha\gamma\delta^{n-1} \\ \alpha\gamma^2\delta^{n-2} \end{array} & \begin{array}{c} \cdots \\ \cdots \end{array} & \begin{array}{c} \alpha\gamma^{n-1}\delta \\ \alpha\gamma^n \end{array} & \begin{array}{c} \beta\delta^m \\ \beta\gamma\delta^{m-1} \end{array} & \begin{array}{c} \cdots \\ \cdots \end{array} & \begin{array}{c} \beta\gamma^{m-1}\delta \\ \beta\gamma^m \end{array} \Bigg) = 1 \\ & YV^1W^{n-1} & YV^2W^{n-2} & & YV^0W^n & ZV^1W^{m-1} & & ZV^mW^0 \end{array}$$

This lead us to consider the homomorphism of $\mathbb{F}_q$-algebras $\Psi : \mathbb{F}_q[\mathbf{X}] \longrightarrow \mathbb{F}_q[Y, Z, V, W]$ given by $\Psi(X_i) = YV^iW^{n-i}$ for $i = 0, \ldots, n$ and $\Psi(X_j) = ZV^{j-n-1}W^{\ell-j}$ for $j = n+1, \ldots, \ell = n+m+1$.

Let $\mathcal{B}$ be the image of Ψ and let's consider $\Psi : \mathbb{F}_q[\mathbf{X}] \longrightarrow \mathcal{B}$.

We can make Ψ a graded morphism of algebras by defining $\deg(Y^yZ^zV^vW^w) = y + z$, and then $\mathcal{B} = \bigoplus_{d \geq 0} \mathcal{B}_d$, where $\mathcal{B}_d$ is generated as an $\mathbb{F}_q$-module by $Y^yZ^zV^vW^w$ with $y + z = d$ and $v + w = ny + mz$.

Codes defined on rational normal scrolls

Thus

$$\begin{aligned} \Psi : \mathbb{F}_q[\mathbf{X}] &\rightarrow \mathcal{B} \\ X_i &\mapsto YV^iW^{n-i} \text{ for } i = 0, \ldots, n \\ X_j &\mapsto ZV^{j-n-1}W^{\ell-j} \text{ for } i = n+1, \ldots, \ell = n+m+1 \end{aligned}$$

is a morphism of graded $\mathbb{F}_q$-algebras. Not surprisingly we proved that $\ker \Psi = I$, proving that $0 \longrightarrow I(d) \longrightarrow \mathbb{F}_q[\mathbf{X}]_d \xrightarrow{\Psi} \mathcal{B}_d \longrightarrow 0$ is an exact sequence of $\mathbb{F}_q$-modules for all $d \geq 0$.

Thus $\mathbb{F}_q[\mathbf{X}]_d/I(d) \cong \mathcal{B}_d$ for all $d \geq 0$ (hence $\mathbb{F}_q[\mathbf{X}]/I \cong \mathcal{B}$) and for some ideal $\mathcal{J} \subset \mathcal{B}$ we must have $\mathbb{F}_q[\mathbf{X}]/I_S \cong \mathcal{B}/\mathcal{J}$. Based on examples like the one we saw (over $\mathbb{F}_5$) we defined $\mathcal{J}$ as a graded ideal $\mathcal{J} = \bigoplus_{d\geq 0} \mathcal{J}(d)$ where $\mathcal{J}(d)$, as an $\mathbb{F}_q$-submodule of $\mathcal{B}_d$, is generated by the elements $Y^yZ^zV^vW^w - Y^{\tilde{y}}Z^{\tilde{z}}V^{\tilde{v}}W^{\tilde{w}}$ where $y, z, v, w, \tilde{y}, \tilde{z}, \tilde{v}, \tilde{w}$ are non-negative integers satisfying

$y + z = \tilde{y} + \tilde{z} = d$	$y = 0 \iff \tilde{y} = 0$
$v + w = ny + mz, \ \tilde{v} + \tilde{w} = n\tilde{y} + m\tilde{z}$	$y = d \iff \tilde{y} = d$
$q - 1 \mid y - \tilde{y}, \ q - 1 \mid v - \tilde{v}$	$v = 0 \iff \tilde{v} = 0$
$v = ny + mz \iff \tilde{v} = n\tilde{y} + m\tilde{z}$	

Codes defined on rational normal scrolls

Thus

$$\begin{array}{rcl} \Psi : \mathbb{F}_q[\mathbf{X}] & \to & \mathcal{B} \\ X_i & \mapsto & YV^iW^{n-i} \text{ for } i = 0, \ldots, n \\ X_j & \mapsto & ZV^{j-n-1}W^{\ell-j} \text{ for } i = n+1, \ldots, \ell = n+m+1 \end{array}$$

is a morphism of graded $\mathbb{F}_q$-algebras. Not surprisingly we proved that $\ker \Psi = I$, proving that $0 \longrightarrow I(d) \longrightarrow \mathbb{F}_q[\mathbf{X}]_d \xrightarrow{\Psi} \mathcal{B}_d \longrightarrow 0$ is an exact sequence of $\mathbb{F}_q$-modules for all $d \geq 0$.

Thus $\mathbb{F}_q[\mathbf{X}]_d/I(d) \cong \mathcal{B}_d$ for all $d \geq 0$ (hence $\mathbb{F}_q[\mathbf{X}]/I \cong \mathcal{B}$) and for some ideal $\mathcal{J} \subset \mathcal{B}$ we must have $\mathbb{F}_q[\mathbf{X}]/I_S \cong \mathcal{B}/\mathcal{J}$. Based on examples like the one we saw (over $\mathbb{F}_5$) we defined $\mathcal{J}$ as a graded ideal $\mathcal{J} = \bigoplus_{d \geq 0} \mathcal{J}(d)$ where $\mathcal{J}(d)$, as an $\mathbb{F}_q$-submodule of $\mathcal{B}_d$, is generated by the elements $Y^yZ^zV^vW^w - Y^{\tilde{y}}Z^{\tilde{z}}V^{\tilde{v}}W^{\tilde{w}}$ where $y, z, v, w, \tilde{y}, \tilde{z}, \tilde{v}, \tilde{w}$ are non-negative integers satisfying

$$\left.\begin{array}{l} y + z = \tilde{y} + \tilde{z} = d \\ v + w = ny + mz, \ \tilde{v} + \tilde{w} = n\tilde{y} + m\tilde{z} \\ q - 1 \mid y - \tilde{y}, \ q - 1 \mid v - \tilde{v} \\ v = ny + mz \iff \tilde{v} = n\tilde{y} + m\tilde{z} \end{array}\right| \begin{array}{c} y = 0 \iff \tilde{y} = 0 \\ y = d \iff \tilde{y} = d \\ v = 0 \iff \tilde{v} = 0 \\ \end{array}$$

Codes defined on rational normal scrolls

Thus

$$\begin{array}{rcl} \Psi : \mathbb{F}_q[\mathbf{X}] & \to & \mathcal{B} \\ X_i & \mapsto & YV^iW^{n-i} \text{ for } i = 0, \ldots, n \\ X_j & \mapsto & ZV^{j-n-1}W^{\ell-j} \text{ for } i = n+1, \ldots, \ell = n+m+1 \end{array}$$

is a morphism of graded $\mathbb{F}_q$-algebras. Not surprisingly we proved that $\ker \Psi = I$, proving that $0 \longrightarrow I(d) \longrightarrow \mathbb{F}_q[\mathbf{X}]_d \xrightarrow{\Psi} \mathcal{B}_d \longrightarrow 0$ is an exact sequence of $\mathbb{F}_q$-modules for all $d \geq 0$.

Thus $\mathbb{F}_q[\mathbf{X}]_d/I(d) \cong \mathcal{B}_d$ for all $d \geq 0$ (hence $\mathbb{F}_q[\mathbf{X}]/I \cong \mathcal{B}$) and for some ideal $\mathcal{J} \subset \mathcal{B}$ we must have $\mathbb{F}_q[\mathbf{X}]/I_S \cong \mathcal{B}/\mathcal{J}$. Based on examples like the one we saw (over $\mathbb{F}_5$) we defined $\mathcal{J}$ as a graded ideal $\mathcal{J} = \bigoplus_{d \geq 0} \mathcal{J}(d)$ where $\mathcal{J}(d)$, as an $\mathbb{F}_q$-submodule of $\mathcal{B}_d$, is generated by the elements $Y^yZ^zV^vW^w - Y^{\tilde{y}}Z^{\tilde{z}}V^{\tilde{v}}W^{\tilde{w}}$ where $y, z, v, w, \tilde{y}, \tilde{z}, \tilde{v}, \tilde{w}$ are non-negative integers satisfying

$$\begin{array}{l|l} y + z = \tilde{y} + \tilde{z} = d & y = 0 \iff \tilde{y} = 0 \\ v + w = ny + mz,\ \tilde{v} + \tilde{w} = n\tilde{y} + m\tilde{z} & y = d \iff \tilde{y} = d \\ q - 1 \mid y - \tilde{y},\ q - 1 \mid v - \tilde{v} & v = 0 \iff \tilde{v} = 0 \\ v = ny + mz \iff \tilde{v} = n\tilde{y} + m\tilde{z} & \end{array}$$

Codes defined on rational normal scrolls

Thus

$$\begin{array}{rcl} \Psi : \mathbb{F}_q[\mathbf{X}] & \to & \mathcal{B} \\ X_i & \mapsto & YV^iW^{n-i} \text{ for } i = 0, \ldots, n \\ X_j & \mapsto & ZV^{j-n-1}W^{\ell-j} \text{ for } i = n+1, \ldots, \ell = n+m+1 \end{array}$$

is a morphism of graded $\mathbb{F}_q$-algebras. Not surprisingly we proved that $\ker \Psi = I$, proving that $0 \longrightarrow I(d) \longrightarrow \mathbb{F}_q[\mathbf{X}]_d \xrightarrow{\Psi} \mathcal{B}_d \longrightarrow 0$ is an exact sequence of $\mathbb{F}_q$-modules for all $d \geq 0$.

Thus $\mathbb{F}_q[\mathbf{X}]_d/I(d) \cong \mathcal{B}_d$ for all $d \geq 0$ (hence $\mathbb{F}_q[\mathbf{X}]/I \cong \mathcal{B}$) and for some ideal $\mathcal{J} \subset \mathcal{B}$ we must have $\mathbb{F}_q[\mathbf{X}]/I_S \cong \mathcal{B}/\mathcal{J}$. Based on examples like the one we saw (over $\mathbb{F}_5$) we defined $\mathcal{J}$ as a graded ideal $\mathcal{J} = \bigoplus_{d \geq 0} \mathcal{J}(d)$ where $\mathcal{J}(d)$, as an $\mathbb{F}_q$-submodule of $\mathcal{B}_d$, is generated by the elements $Y^yZ^zV^vW^w - Y^{\tilde{y}}Z^{\tilde{z}}V^{\tilde{v}}W^{\tilde{w}}$ where $y, z, v, w, \tilde{y}, \tilde{z}, \tilde{v}, \tilde{w}$ are non-negative integers satisfying

$$\left.\begin{array}{l} y + z = \tilde{y} + \tilde{z} = d \\ v + w = ny + mz, \ \tilde{v} + \tilde{w} = n\tilde{y} + m\tilde{z} \\ q-1 \mid y - \tilde{y}, \ q-1 \mid v - \tilde{v} \\ v = ny + mz \iff \tilde{v} = n\tilde{y} + m\tilde{z} \end{array}\right| \begin{array}{l} y = 0 \iff \tilde{y} = 0 \\ y = d \iff \tilde{y} = d \\ v = 0 \iff \tilde{v} = 0 \\ \ \end{array}$$

Codes defined on rational normal scrolls

Thus

$$\begin{aligned} \Psi : \mathbb{F}_q[\mathbf{X}] &\to \mathcal{B} \\ X_i &\mapsto YV^iW^{n-i} \text{ for } i = 0, \ldots, n \\ X_j &\mapsto ZV^{j-n-1}W^{\ell-j} \text{ for } i = n+1, \ldots, \ell = n+m+1 \end{aligned}$$

is a morphism of graded $\mathbb{F}_q$-algebras. Not surprisingly we proved that $\ker \Psi = I$, proving that $0 \longrightarrow I(d) \longrightarrow \mathbb{F}_q[\mathbf{X}]_d \xrightarrow{\Psi} \mathcal{B}_d \longrightarrow 0$ is an exact sequence of $\mathbb{F}_q$-modules for all $d \geq 0$.

Thus $\mathbb{F}_q[\mathbf{X}]_d/I(d) \cong \mathcal{B}_d$ for all $d \geq 0$ (hence $\mathbb{F}_q[\mathbf{X}]/I \cong \mathcal{B}$) and for some ideal $\mathcal{J} \subset \mathcal{B}$ we must have $\mathbb{F}_q[\mathbf{X}]/I_S \cong \mathcal{B}/\mathcal{J}$. Based on examples like the one we saw (over $\mathbb{F}_5$) we defined $\mathcal{J}$ as a graded ideal $\mathcal{J} = \bigoplus_{d \geq 0} \mathcal{J}(d)$ where $\mathcal{J}(d)$, as an $\mathbb{F}_q$-submodule of $\mathcal{B}_d$, is generated by the elements $Y^yZ^zV^vW^w - Y^{\tilde{y}}Z^{\tilde{z}}V^{\tilde{v}}W^{\tilde{w}}$ where $y, z, v, w, \tilde{y}, \tilde{z}, \tilde{v}, \tilde{w}$ are non-negative integers satisfying

$y + z = \tilde{y} + \tilde{z} = d$	$y = 0 \iff \tilde{y} = 0$
$v + w = ny + mz,\ \tilde{v} + \tilde{w} = n\tilde{y} + m\tilde{z}$	$y = d \iff \tilde{y} = d$
$q - 1 \mid y - \tilde{y},\ q - 1 \mid v - \tilde{v}$	$v = 0 \iff \tilde{v} = 0$
$v = ny + mz \iff \tilde{v} = n\tilde{y} + m\tilde{z}$	

Codes defined on rational normal scrolls

Thus

$$\begin{array}{rcl} \Psi : \mathbb{F}_q[\mathbf{X}] & \to & \mathcal{B} \\ X_i & \mapsto & YV^iW^{n-i} \text{ for } i = 0, \dots, n \\ X_j & \mapsto & ZV^{j-n-1}W^{\ell-j} \text{ for } i = n+1, \dots, \ell = n+m+1 \end{array}$$

is a morphism of graded $\mathbb{F}_q$-algebras. Not surprisingly we proved that $\ker \Psi = I$, proving that $0 \longrightarrow I(d) \longrightarrow \mathbb{F}_q[\mathbf{X}]_d \xrightarrow{\Psi} \mathcal{B}_d \longrightarrow 0$ is an exact sequence of $\mathbb{F}_q$-modules for all $d \geq 0$.

Thus $\mathbb{F}_q[\mathbf{X}]_d/I(d) \cong \mathcal{B}_d$ for all $d \geq 0$ (hence $\mathbb{F}_q[\mathbf{X}]/I \cong \mathcal{B}$) and for some ideal $\mathcal{J} \subset \mathcal{B}$ we must have $\mathbb{F}_q[\mathbf{X}]/I_S \cong \mathcal{B}/\mathcal{J}$. Based on examples like the one we saw (over $\mathbb{F}_5$) we defined $\mathcal{J}$ as a graded ideal $\mathcal{J} = \bigoplus_{d \geq 0} \mathcal{J}(d)$ where $\mathcal{J}(d)$, as an $\mathbb{F}_q$-submodule of $\mathcal{B}_d$, is generated by the elements $Y^yZ^zV^vW^w - Y^{\tilde{y}}Z^{\tilde{z}}V^{\tilde{v}}W^{\tilde{w}}$ where $y, z, v, w, \tilde{y}, \tilde{z}, \tilde{v}, \tilde{w}$ are non-negative integers satisfying

$$\left.\begin{array}{l} y + z = \tilde{y} + \tilde{z} = d \\ v + w = ny + mz, \ \tilde{v} + \tilde{w} = n\tilde{y} + m\tilde{z} \\ q - 1 \mid y - \tilde{y}, \ q - 1 \mid v - \tilde{v} \\ v = ny + mz \iff \tilde{v} = n\tilde{y} + m\tilde{z} \end{array}\right| \begin{array}{l} y = 0 \iff \tilde{y} = 0 \\ y = d \iff \tilde{y} = d \\ v = 0 \iff \tilde{v} = 0 \\ \ \end{array}$$

Codes defined on rational normal scrolls

Thus

$$\begin{array}{rcl} \Psi : \mathbb{F}_q[\mathbf{X}] & \to & \mathcal{B} \\ X_i & \mapsto & YV^iW^{n-i} \text{ for } i = 0, \ldots, n \\ X_j & \mapsto & ZV^{j-n-1}W^{\ell-j} \text{ for } i = n+1, \ldots, \ell = n+m+1 \end{array}$$

is a morphism of graded $\mathbb{F}_q$-algebras. Not surprisingly we proved that $\ker \Psi = I$, proving that $0 \longrightarrow I(d) \longrightarrow \mathbb{F}_q[\mathbf{X}]_d \xrightarrow{\Psi} \mathcal{B}_d \longrightarrow 0$ is an exact sequence of $\mathbb{F}_q$-modules for all $d \geq 0$.

Thus $\mathbb{F}_q[\mathbf{X}]_d/I(d) \cong \mathcal{B}_d$ for all $d \geq 0$ (hence $\mathbb{F}_q[\mathbf{X}]/I \cong \mathcal{B}$) and for some ideal $\mathcal{J} \subset \mathcal{B}$ we must have $\mathbb{F}_q[\mathbf{X}]/I_S \cong \mathcal{B}/\mathcal{J}$. Based on examples like the one we saw (over $\mathbb{F}_5$) we defined $\mathcal{J}$ as a graded ideal $\mathcal{J} = \bigoplus_{d \geq 0} \mathcal{J}(d)$ where $\mathcal{J}(d)$, as an $\mathbb{F}_q$-submodule of $\mathcal{B}_d$, is generated by the elements $Y^yZ^zV^vW^w - Y^{\tilde{y}}Z^{\tilde{z}}V^{\tilde{v}}W^{\tilde{w}}$ where $y, z, v, w, \tilde{y}, \tilde{z}, \tilde{v}, \tilde{w}$ are non-negative integers satisfying

$$\left.\begin{array}{l} y + z = \tilde{y} + \tilde{z} = d \\ v + w = ny + mz, \ \tilde{v} + \tilde{w} = n\tilde{y} + m\tilde{z} \\ q-1 \mid y - \tilde{y}, \ q-1 \mid v - \tilde{v} \\ v = ny + mz \iff \tilde{v} = n\tilde{y} + m\tilde{z} \end{array}\right| \begin{array}{l} y = 0 \iff \tilde{y} = 0 \\ y = d \iff \tilde{y} = d \\ v = 0 \iff \tilde{v} = 0 \\ \end{array}$$

Codes defined on rational normal scrolls

Thus

$$\begin{array}{rcl} \Psi : \mathbb{F}_q[\mathbf{X}] & \to & \mathcal{B} \\ X_i & \mapsto & YV^iW^{n-i} \text{ for } i = 0, \dots, n \\ X_j & \mapsto & ZV^{j-n-1}W^{\ell-j} \text{ for } i = n+1, \dots, \ell = n+m+1 \end{array}$$

is a morphism of graded $\mathbb{F}_q$-algebras. Not surprisingly we proved that

$\ker \Psi = I$, proving that $0 \longrightarrow I(d) \longrightarrow \mathbb{F}_q[\mathbf{X}]_d \xrightarrow{\Psi} \mathcal{B}_d \longrightarrow 0$ is an exact sequence of $\mathbb{F}_q$-modules for all $d \geq 0$.

Thus $\mathbb{F}_q[\mathbf{X}]_d / I(d) \cong \mathcal{B}_d$ for all $d \geq 0$ (hence $\mathbb{F}_q[\mathbf{X}]/I \cong \mathcal{B}$) and for some ideal $\mathcal{J} \subset \mathcal{B}$ we must have $\mathbb{F}_q[\mathbf{X}]/I_S \cong \mathcal{B}/\mathcal{J}$. Based on examples like the one we saw (over $\mathbb{F}_5$) we defined $\mathcal{J}$ as a graded ideal $\mathcal{J} = \bigoplus_{d \geq 0} \mathcal{J}(d)$ where $\mathcal{J}(d)$, as an $\mathbb{F}_q$-submodule of $\mathcal{B}_d$, is generated by the elements $Y^yZ^zV^vW^w - Y^{\tilde{y}}Z^{\tilde{z}}V^{\tilde{v}}W^{\tilde{w}}$ where $y, z, v, w, \tilde{y}, \tilde{z}, \tilde{v}, \tilde{w}$ are non-negative integers satisfying

$$\begin{array}{l|l} y + z = \tilde{y} + \tilde{z} = d & y = 0 \iff \tilde{y} = 0 \\ v + w = ny + mz,\ \tilde{v} + \tilde{w} = n\tilde{y} + m\tilde{z} & y = d \iff \tilde{y} = d \\ q-1 \mid y - \tilde{y},\ q-1 \mid v - \tilde{v} & v = 0 \iff \tilde{v} = 0 \\ v = ny + mz \iff \tilde{v} = n\tilde{y} + m\tilde{z} & \end{array}$$

Codes defined on rational normal scrolls

Thus

$$\begin{array}{rcl} \Psi : \mathbb{F}_q[\mathbf{X}] & \to & \mathcal{B} \\ X_i & \mapsto & YV^iW^{n-i} \text{ for } i = 0, \ldots, n \\ X_j & \mapsto & ZV^{j-n-1}W^{\ell-j} \text{ for } i = n+1, \ldots, \ell = n+m+1 \end{array}$$

is a morphism of graded $\mathbb{F}_q$-algebras. Not surprisingly we proved that $\ker \Psi = I$, proving that $0 \longrightarrow I(d) \longrightarrow \mathbb{F}_q[\mathbf{X}]_d \xrightarrow{\Psi} \mathcal{B}_d \longrightarrow 0$ is an exact sequence of $\mathbb{F}_q$-modules for all $d \geq 0$.

Thus $\mathbb{F}_q[\mathbf{X}]_d/I(d) \cong \mathcal{B}_d$ for all $d \geq 0$ (hence $\mathbb{F}_q[\mathbf{X}]/I \cong \mathcal{B}$) and for some ideal $\mathcal{J} \subset \mathcal{B}$ we must have $\mathbb{F}_q[\mathbf{X}]/I_S \cong \mathcal{B}/\mathcal{J}$. Based on examples like the one we saw (over $\mathbb{F}_5$) we defined $\mathcal{J}$ as a graded ideal $\mathcal{J} = \bigoplus_{d \geq 0} \mathcal{J}(d)$ where $\mathcal{J}(d)$, as an $\mathbb{F}_q$-submodule of $\mathcal{B}_d$, is generated by the elements $Y^yZ^zV^vW^w - Y^{\tilde{y}}Z^{\tilde{z}}V^{\tilde{v}}W^{\tilde{w}}$ where $y, z, v, w, \tilde{y}, \tilde{z}, \tilde{v}, \tilde{w}$ are non-negative integers satisfying

$$\left.\begin{array}{l} y + z = \tilde{y} + \tilde{z} = d \\ v + w = ny + mz, \ \tilde{v} + \tilde{w} = n\tilde{y} + m\tilde{z} \\ q-1 \mid y - \tilde{y}, \ q-1 \mid v - \tilde{v} \\ v = ny + mz \iff \tilde{v} = n\tilde{y} + m\tilde{z} \end{array}\right| \begin{array}{l} y = 0 \iff \tilde{y} = 0 \\ y = d \iff \tilde{y} = d \\ v = 0 \iff \tilde{v} = 0 \end{array}$$

Codes defined on rational normal scrolls

Thus

$$\begin{array}{rcl} \Psi : \mathbb{F}_q[\mathbf{X}] & \to & \mathcal{B} \\ X_i & \mapsto & YV^iW^{n-i} \text{ for } i = 0, \dots, n \\ X_j & \mapsto & ZV^{j-n-1}W^{\ell-j} \text{ for } i = n+1, \dots, \ell = n+m+1 \end{array}$$

is a morphism of graded $\mathbb{F}_q$-algebras. Not surprisingly we proved that $\ker \Psi = I$, proving that $0 \longrightarrow I(d) \longrightarrow \mathbb{F}_q[\mathbf{X}]_d \xrightarrow{\Psi} \mathcal{B}_d \longrightarrow 0$ is an exact sequence of $\mathbb{F}_q$-modules for all $d \geq 0$.

Thus $\mathbb{F}_q[\mathbf{X}]_d/I(d) \cong \mathcal{B}_d$ for all $d \geq 0$ (hence $\mathbb{F}_q[\mathbf{X}]/I \cong \mathcal{B}$) and for some ideal $\mathcal{J} \subset \mathcal{B}$ we must have $\mathbb{F}_q[\mathbf{X}]/I_S \cong \mathcal{B}/\mathcal{J}$. Based on examples like the one we saw (over $\mathbb{F}_5$) we defined $\mathcal{J}$ as a graded ideal $\mathcal{J} = \bigoplus_{d \geq 0} \mathcal{J}(d)$ where $\mathcal{J}(d)$, as an $\mathbb{F}_q$-submodule of $\mathcal{B}_d$, is generated by the elements $Y^yZ^zV^vW^w - Y^{\tilde{y}}Z^{\tilde{z}}V^{\tilde{v}}W^{\tilde{w}}$ where $y, z, v, w, \tilde{y}, \tilde{z}, \tilde{v}, \tilde{w}$ are non-negative integers satisfying

$$\begin{array}{l|l} y + z = \tilde{y} + \tilde{z} = d & y = 0 \iff \tilde{y} = 0 \\ v + w = ny + mz,\ \tilde{v} + \tilde{w} = n\tilde{y} + m\tilde{z} & y = d \iff \tilde{y} = d \\ q-1 \mid y - \tilde{y},\ q-1 \mid v - \tilde{v} & v = 0 \iff \tilde{v} = 0 \\ v = ny + mz \iff \tilde{v} = n\tilde{y} + m\tilde{z} & \end{array}$$

Codes defined on rational normal scrolls

Thus

$$\begin{array}{rcl} \Psi : \mathbb{F}_q[\mathbf{X}] & \to & \mathcal{B} \\ X_i & \mapsto & YV^iW^{n-i} \text{ for } i = 0, \ldots, n \\ X_j & \mapsto & ZV^{j-n-1}W^{\ell-j} \text{ for } i = n+1, \ldots, \ell = n+m+1 \end{array}$$

is a morphism of graded $\mathbb{F}_q$-algebras. Not surprisingly we proved that

$\ker \Psi = I$, proving that $0 \longrightarrow I(d) \longrightarrow \mathbb{F}_q[\mathbf{X}]_d \xrightarrow{\Psi} \mathcal{B}_d \longrightarrow 0$ is an exact sequence of $\mathbb{F}_q$-modules for all $d \geq 0$.

Thus $\mathbb{F}_q[\mathbf{X}]_d/I(d) \cong \mathcal{B}_d$ for all $d \geq 0$ (hence $\mathbb{F}_q[\mathbf{X}]/I \cong \mathcal{B}$) and for some ideal $\mathcal{J} \subset \mathcal{B}$ we must have $\mathbb{F}_q[\mathbf{X}]/I_S \cong \mathcal{B}/\mathcal{J}$. Based on examples like the one we saw (over $\mathbb{F}_5$) we defined $\mathcal{J}$ as a graded ideal $\mathcal{J} = \bigoplus_{d \geq 0} \mathcal{J}(d)$ where $\mathcal{J}(d)$, as an $\mathbb{F}_q$-submodule of $\mathcal{B}_d$, is generated by the elements $Y^yZ^zV^vW^w - Y^{\tilde{y}}Z^{\tilde{z}}V^{\tilde{v}}W^{\tilde{w}}$ where $y, z, v, w, \tilde{y}, \tilde{z}, \tilde{v}, \tilde{w}$ are non-negative integers satisfying

$$\begin{array}{l|l} y + z = \tilde{y} + \tilde{z} = d & y = 0 \iff \tilde{y} = 0 \\ v + w = ny + mz, \ \tilde{v} + \tilde{w} = n\tilde{y} + m\tilde{z} & y = d \iff \tilde{y} = d \\ q - 1 \mid y - \tilde{y}, \ q - 1 \mid v - \tilde{v} & v = 0 \iff \tilde{v} = 0 \\ v = ny + mz \iff \tilde{v} = n\tilde{y} + m\tilde{z} & \end{array}$$

Codes defined on rational normal scrolls

Thus

$$\begin{array}{rcl} \Psi : \mathbb{F}_q[\mathbf{X}] & \to & \mathcal{B} \\ X_i & \mapsto & YV^iW^{n-i} \text{ for } i = 0, \ldots, n \\ X_j & \mapsto & ZV^{j-n-1}W^{\ell-j} \text{ for } i = n+1, \ldots, \ell = n+m+1 \end{array}$$

is a morphism of graded $\mathbb{F}_q$-algebras. Not surprisingly we proved that $\ker \Psi = I$, proving that $0 \longrightarrow I(d) \longrightarrow \mathbb{F}_q[\mathbf{X}]_d \xrightarrow{\Psi} \mathcal{B}_d \longrightarrow 0$ is an exact sequence of $\mathbb{F}_q$-modules for all $d \geq 0$.

Thus $\mathbb{F}_q[\mathbf{X}]_d/I(d) \cong \mathcal{B}_d$ for all $d \geq 0$ (hence $\mathbb{F}_q[\mathbf{X}]/I \cong \mathcal{B}$) and for some ideal $\mathcal{J} \subset \mathcal{B}$ we must have $\mathbb{F}_q[\mathbf{X}]/I_S \cong \mathcal{B}/\mathcal{J}$. Based on examples like the one we saw (over $\mathbb{F}_5$) we defined $\mathcal{J}$ as a graded ideal $\mathcal{J} = \bigoplus_{d \geq 0} \mathcal{J}(d)$ where $\mathcal{J}(d)$, as an $\mathbb{F}_q$-submodule of $\mathcal{B}_d$, is generated by the elements $Y^yZ^zV^vW^w - Y^{\tilde{y}}Z^{\tilde{z}}V^{\tilde{v}}W^{\tilde{w}}$ where $y, z, v, w, \tilde{y}, \tilde{z}, \tilde{v}, \tilde{w}$ are non-negative integers satisfying

$$\begin{array}{l|l} y + z = \tilde{y} + \tilde{z} = d & y = 0 \iff \tilde{y} = 0 \\ v + w = ny + mz, \ \tilde{v} + \tilde{w} = n\tilde{y} + m\tilde{z} & y = d \iff \tilde{y} = d \\ q-1 \mid y - \tilde{y}, \ q-1 \mid v - \tilde{v} & v = 0 \iff \tilde{v} = 0 \\ v = ny + mz \iff \tilde{v} = n\tilde{y} + m\tilde{z} & \end{array}$$

Codes defined on rational normal scrolls

Thus

$$\begin{array}{rcl} \Psi : \mathbb{F}_q[\mathbf{X}] & \to & \mathcal{B} \\ X_i & \mapsto & YV^iW^{n-i} \text{ for } i = 0, \ldots, n \\ X_j & \mapsto & ZV^{j-n-1}W^{\ell-j} \text{ for } i = n+1, \ldots, \ell = n+m+1 \end{array}$$

is a morphism of graded $\mathbb{F}_q$-algebras. Not surprisingly we proved that $\ker \Psi = I$, proving that $0 \longrightarrow I(d) \longrightarrow \mathbb{F}_q[\mathbf{X}]_d \xrightarrow{\Psi} \mathcal{B}_d \longrightarrow 0$ is an exact sequence of $\mathbb{F}_q$-modules for all $d \geq 0$.

Thus $\mathbb{F}_q[\mathbf{X}]_d/I(d) \cong \mathcal{B}_d$ for all $d \geq 0$ (hence $\mathbb{F}_q[\mathbf{X}]/I \cong \mathcal{B}$) and for some ideal $\mathcal{J} \subset \mathcal{B}$ we must have $\mathbb{F}_q[\mathbf{X}]/I_S \cong \mathcal{B}/\mathcal{J}$. Based on examples like the one we saw (over $\mathbb{F}_5$) we defined $\mathcal{J}$ as a graded ideal $\mathcal{J} = \bigoplus_{d \geq 0} \mathcal{J}(d)$ where $\mathcal{J}(d)$, as an $\mathbb{F}_q$-submodule of $\mathcal{B}_d$, is generated by the elements $Y^yZ^zV^vW^w - Y^{\tilde{y}}Z^{\tilde{z}}V^{\tilde{v}}W^{\tilde{w}}$ where $y, z, v, w, \tilde{y}, \tilde{z}, \tilde{v}, \tilde{w}$ are non-negative integers satisfying

$$\left.\begin{array}{l} y + z = \tilde{y} + \tilde{z} = d \\ v + w = ny + mz, \ \tilde{v} + \tilde{w} = n\tilde{y} + m\tilde{z} \\ q - 1 \mid y - \tilde{y}, \ q - 1 \mid v - \tilde{v} \\ v = ny + mz \Longleftrightarrow \tilde{v} = n\tilde{y} + m\tilde{z} \end{array} \right| \begin{array}{l} y = 0 \Longleftrightarrow \tilde{y} = 0 \\ y = d \Longleftrightarrow \tilde{y} = d \\ v = 0 \Longleftrightarrow \tilde{v} = 0 \\ \end{array}$$

Codes defined on rational normal scrolls

After some work we proved that the sequence
$0 \longrightarrow I_S(d) \longrightarrow \mathbb{F}_q[\mathbf{X}]_d \xrightarrow{\Psi} \mathcal{B}_d/\mathcal{J}(d) \longrightarrow 0$ is an exact sequence of $\mathbb{F}_q$-modules for all $d \geq 0$. So
$\dim_{\mathbb{F}_q} C_d = \dim_{\mathbb{F}_q} \mathbb{F}_q[\mathbf{X}]_d / I_S(d) = \dim_{\mathbb{F}_q} \mathcal{B}_d/\mathcal{J}(d)$.

We defined a order on the monomials of $\mathcal{B}$ and proved that if we define $\Delta(\mathcal{J})$ to be the set of monomials which are not leading monomials of $\mathcal{J}$ (with respect to that order) then the classes of the monomials of $\Delta(\mathcal{J})$ in $\mathcal{B}/\mathcal{J}$ form a basis for $\mathcal{B}/\mathcal{J}$ as an $\mathbb{F}_q$-vector space.

Moreover, the classes of the monomials of degree d which are not leading monomials of polynomials in $\mathcal{J}(d)$ form a basis as an $\mathbb{F}_q$-vector space basis for $\mathcal{B}_d/\mathcal{J}(d)$.

Codes defined on rational normal scrolls

After some work we proved that the sequence
$0 \longrightarrow I_S(d) \longrightarrow \mathbb{F}_q[\mathbf{X}]_d \xrightarrow{\Psi} \mathcal{B}_d/\mathcal{J}(d) \longrightarrow 0$ is an exact sequence of $\mathbb{F}_q$-modules for all $d \geq 0$. So
$\dim_{\mathbb{F}_q} C_d = \dim_{\mathbb{F}_q} \mathbb{F}_q[\mathbf{X}]_d/I_S(d) = \dim_{\mathbb{F}_q} \mathcal{B}_d/\mathcal{J}(d)$.

We defined a order on the monomials of $\mathcal{B}$ and proved that if we define $\Delta(\mathcal{J})$ to be the set of monomials which are not leading monomials of $\mathcal{J}$ (with respect to that order) then the classes of the monomials of $\Delta(\mathcal{J})$ in $\mathcal{B}/\mathcal{J}$ form a basis for $\mathcal{B}/\mathcal{J}$ as an $\mathbb{F}_q$-vector space.

Moreover, the classes of the monomials of degree d which are not leading monomials of polynomials in $\mathcal{J}(d)$ form a basis as an $\mathbb{F}_q$-vector space basis for $\mathcal{B}_d/\mathcal{J}(d)$.

Codes defined on rational normal scrolls

After some work we proved that the sequence
$0 \longrightarrow I_S(d) \longrightarrow \mathbb{F}_q[\mathbf{X}]_d \xrightarrow{\Psi} \mathcal{B}_d/\mathcal{J}(d) \longrightarrow 0$ is an exact sequence of $\mathbb{F}_q$-modules for all $d \geq 0$. So
$\dim_{\mathbb{F}_q} C_d = \dim_{\mathbb{F}_q} \mathbb{F}_q[\mathbf{X}]_d / I_S(d) = \dim_{\mathbb{F}_q} \mathcal{B}_d/\mathcal{J}(d)$.

We defined a order on the monomials of $\mathcal{B}$ and proved that if we define $\Delta(\mathcal{J})$ to be the set of monomials which are not leading monomials of $\mathcal{J}$ (with respect to that order) then the classes of the monomials of $\Delta(\mathcal{J})$ in $\mathcal{B}/\mathcal{J}$ form a basis for $\mathcal{B}/\mathcal{J}$ as an $\mathbb{F}_q$-vector space.

Moreover, the classes of the monomials of degree d which are not leading monomials of polynomials in $\mathcal{J}(d)$ form a basis as an $\mathbb{F}_q$-vector space basis for $\mathcal{B}_d/\mathcal{J}(d)$.

Codes defined on rational normal scrolls

After some work we proved that the sequence
$0 \longrightarrow I_S(d) \longrightarrow \mathbb{F}_q[\mathbf{X}]_d \xrightarrow{\Psi} \mathcal{B}_d/\mathcal{J}(d) \longrightarrow 0$ is an exact sequence of $\mathbb{F}_q$-modules for all $d \geq 0$. So
$\dim_{\mathbb{F}_q} C_d = \dim_{\mathbb{F}_q} \mathbb{F}_q[\mathbf{X}]_d / I_S(d) = \dim_{\mathbb{F}_q} \mathcal{B}_d/\mathcal{J}(d)$.

We defined a order on the monomials of $\mathcal{B}$ and proved that if we define $\Delta(\mathcal{J})$ to be the set of monomials which are not leading monomials of $\mathcal{J}$ (with respect to that order) then the classes of the monomials of $\Delta(\mathcal{J})$ in $\mathcal{B}/\mathcal{J}$ form a basis for $\mathcal{B}/\mathcal{J}$ as an $\mathbb{F}_q$-vector space.

Moreover, the classes of the monomials of degree d which are not leading monomials of polynomials in $\mathcal{J}(d)$ form a basis as an $\mathbb{F}_q$-vector space basis for $\mathcal{B}_d/\mathcal{J}(d)$.

Codes defined on rational normal scrolls

After some work we proved that the sequence
$0 \longrightarrow I_S(d) \longrightarrow \mathbb{F}_q[\mathbf{X}]_d \xrightarrow{\Psi} \mathcal{B}_d/\mathcal{J}(d) \longrightarrow 0$ is an exact sequence of $\mathbb{F}_q$-modules for all $d \geq 0$. So
$\dim_{\mathbb{F}_q} C_d = \dim_{\mathbb{F}_q} \mathbb{F}_q[\mathbf{X}]_d/I_S(d) = \dim_{\mathbb{F}_q} \mathcal{B}_d/\mathcal{J}(d)$.

We defined a order on the monomials of $\mathcal{B}$ and proved that if we define $\Delta(\mathcal{J})$ to be the set of monomials which are not leading monomials of $\mathcal{J}$ (with respect to that order) then the classes of the monomials of $\Delta(\mathcal{J})$ in $\mathcal{B}/\mathcal{J}$ form a basis for $\mathcal{B}/\mathcal{J}$ as an $\mathbb{F}_q$-vector space.

Moreover, the classes of the monomials of degree d which are not leading monomials of polynomials in $\mathcal{J}(d)$ form a basis as an $\mathbb{F}_q$-vector space basis for $\mathcal{B}_d/\mathcal{J}(d)$.

Codes defined on rational normal scrolls

After some work we proved that the sequence $0 \longrightarrow I_S(d) \longrightarrow \mathbb{F}_q[\mathbf{X}]_d \xrightarrow{\Psi} \mathcal{B}_d/\mathcal{J}(d) \longrightarrow 0$ is an exact sequence of $\mathbb{F}_q$-modules for all $d \geq 0$. So $\dim_{\mathbb{F}_q} C_d = \dim_{\mathbb{F}_q} \mathbb{F}_q[\mathbf{X}]_d / I_S(d) = \dim_{\mathbb{F}_q} \mathcal{B}_d/\mathcal{J}(d)$.

We defined a order on the monomials of $\mathcal{B}$ and proved that if we define $\Delta(\mathcal{J})$ to be the set of monomials which are not leading monomials of $\mathcal{J}$ (with respect to that order) then the classes of the monomials of $\Delta(\mathcal{J})$ in $\mathcal{B}/\mathcal{J}$ form a basis for $\mathcal{B}/\mathcal{J}$ as an $\mathbb{F}_q$-vector space.

Moreover, the classes of the monomials of degree d which are not leading monomials of polynomials in $\mathcal{J}(d)$ form a basis as an $\mathbb{F}_q$-vector space basis for $\mathcal{B}_d/\mathcal{J}(d)$.

Codes defined on rational normal scrolls

After some work we proved that the sequence
$0 \longrightarrow I_S(d) \longrightarrow \mathbb{F}_q[\mathbf{X}]_d \xrightarrow{\Psi} \mathcal{B}_d/\mathcal{J}(d) \longrightarrow 0$ is an exact sequence of $\mathbb{F}_q$-modules for all $d \geq 0$. So
$\dim_{\mathbb{F}_q} C_d = \dim_{\mathbb{F}_q} \mathbb{F}_q[\mathbf{X}]_d/I_S(d) = \dim_{\mathbb{F}_q} \mathcal{B}_d/\mathcal{J}(d)$.

We defined a order on the monomials of $\mathcal{B}$ and proved that if we define $\Delta(\mathcal{J})$ to be the set of monomials which are not leading monomials of $\mathcal{J}$ (with respect to that order) then the classes of the monomials of $\Delta(\mathcal{J})$ in $\mathcal{B}/\mathcal{J}$ form a basis for $\mathcal{B}/\mathcal{J}$ as an $\mathbb{F}_q$-vector space.

Moreover, the classes of the monomials of degree d which are not leading monomials of polynomials in $\mathcal{J}(d)$ form a basis as an $\mathbb{F}_q$-vector space basis for $\mathcal{B}_d/\mathcal{J}(d)$.

Codes defined on rational normal scrolls

After some work we proved that the sequence
$0 \longrightarrow I_S(d) \longrightarrow \mathbb{F}_q[\mathbf{X}]_d \xrightarrow{\Psi} \mathcal{B}_d/\mathcal{J}(d) \longrightarrow 0$ is an exact sequence of $\mathbb{F}_q$-modules for all $d \geq 0$. So
$\dim_{\mathbb{F}_q} C_d = \dim_{\mathbb{F}_q} \mathbb{F}_q[\mathbf{X}]_d / I_S(d) = \dim_{\mathbb{F}_q} \mathcal{B}_d/\mathcal{J}(d)$.

We defined a order on the monomials of $\mathcal{B}$ and proved that if we define $\Delta(\mathcal{J})$ to be the set of monomials which are not leading monomials of $\mathcal{J}$ (with respect to that order) then the classes of the monomials of $\Delta(\mathcal{J})$ in $\mathcal{B}/\mathcal{J}$ form a basis for $\mathcal{B}/\mathcal{J}$ as an $\mathbb{F}_q$-vector space.

Moreover, the classes of the monomials of degree d which are not leading monomials of polynomials in $\mathcal{J}(d)$ form a basis as an $\mathbb{F}_q$-vector space basis for $\mathcal{B}_d/\mathcal{J}(d)$.

Codes defined on rational normal scrolls

After some work we proved that the sequence
$0 \longrightarrow I_S(d) \longrightarrow \mathbb{F}_q[\mathbf{X}]_d \xrightarrow{\Psi} \mathcal{B}_d/\mathcal{J}(d) \longrightarrow 0$ is an exact sequence of $\mathbb{F}_q$-modules for all $d \geq 0$. So
$\dim_{\mathbb{F}_q} C_d = \dim_{\mathbb{F}_q} \mathbb{F}_q[\mathbf{X}]_d/I_S(d) = \dim_{\mathbb{F}_q} \mathcal{B}_d/\mathcal{J}(d)$.

We defined a order on the monomials of $\mathcal{B}$ and proved that if we define $\Delta(\mathcal{J})$ to be the set of monomials which are not leading monomials of $\mathcal{J}$ (with respect to that order) then the classes of the monomials of $\Delta(\mathcal{J})$ in $\mathcal{B}/\mathcal{J}$ form a basis for $\mathcal{B}/\mathcal{J}$ as an $\mathbb{F}_q$-vector space.

Moreover, the classes of the monomials of degree d which are not leading monomials of polynomials in $\mathcal{J}(d)$ form a basis as an $\mathbb{F}_q$-vector space basis for $\mathcal{B}_d/\mathcal{J}(d)$.

Codes defined on rational normal scrolls

We were able to count these classes and prove that the dimension of C_d is:

(a) $(n+m)\dfrac{d(d+1)}{2} + d + 1$ if $d \leq q/n$;

(b) $(s+1)\left[\dfrac{(n+m)s}{2} + m(d-s) + 1\right] + (d-s)(q+1)$

if $m < n$ and $q/n < d \leq q/m$, where $s = \left\lfloor \dfrac{q-md}{n-m} \right\rfloor$;

(c) $(d+1)(q+1)$ if $q/m < d < q$;

(d) $(q+1)^2$ if $q \leq d$.

Observe that if $d \geq q$ then the evaluation morphism $\varphi : \mathbb{F}[\mathbf{X}]_d / I_S(d) \to \mathbb{F}_q^N$, where $N = (q+1)^2$, is surjective, so the code is equal to $\mathbb{F}_q^{(q+1)^2}$ and the minimum distance is equal to 1.

Codes defined on rational normal scrolls

We were able to count these classes and prove that the dimension of C_d is:

(a) $(n+m)\dfrac{d(d+1)}{2} + d + 1$ if $d \leq q/n$;

(b) $(s+1)\left[\dfrac{(n+m)s}{2} + m(d-s) + 1\right] + (d-s)(q+1)$
if $m < n$ and $q/n < d \leq q/m$, where $s = \left\lfloor \dfrac{q-md}{n-m} \right\rfloor$;

(c) $(d+1)(q+1)$ if $q/m < d < q$;

(d) $(q+1)^2$ if $q \leq d$.

Observe that if $d \geq q$ then the evaluation morphism $\varphi : \mathbb{F}[\mathbf{X}]_d / I_S(d) \to \mathbb{F}_q^N$, where $N = (q+1)^2$, is surjective, so the code is equal to $\mathbb{F}_q^{(q+1)^2}$ and the minimum distance is equal to 1.

Codes defined on rational normal scrolls

We were able to count these classes and prove that the dimension of C_d is:

(a) $(n+m)\dfrac{d(d+1)}{2} + d + 1$ if $d \leq q/n$;

(b) $(s+1)\left[\dfrac{(n+m)s}{2} + m(d-s) + 1\right] + (d-s)(q+1)$
if $m < n$ and $q/n < d \leq q/m$, where $s = \left\lfloor \dfrac{q-md}{n-m} \right\rfloor$;

(c) $(d+1)(q+1)$ if $q/m < d < q$;

(d) $(q+1)^2$ if $q \leq d$.

Observe that if $d \geq q$ then the evaluation morphism $\varphi : \mathbb{F}[\mathbf{X}]_d / I_S(d) \to \mathbb{F}_q^N$, where $N = (q+1)^2$, is surjective, so the code is equal to $\mathbb{F}_q^{(q+1)^2}$ and the minimum distance is equal to 1.

Codes defined on rational normal scrolls

We were able to count these classes and prove that the dimension of C_d is:

(a) $(n+m)\dfrac{d(d+1)}{2} + d + 1$ if $d \leq q/n$;

(b) $(s+1)\left[\dfrac{(n+m)s}{2} + m(d-s) + 1\right] + (d-s)(q+1)$
 if $m < n$ and $q/n < d \leq q/m$, where $s = \left\lfloor \dfrac{q-md}{n-m} \right\rfloor$;

(c) $(d+1)(q+1)$ if $q/m < d < q$;

(d) $(q+1)^2$ if $q \leq d$.

Observe that if $d \geq q$ then the evaluation morphism $\varphi : \mathbb{F}[\mathbf{X}]_d / I_S(d) \to \mathbb{F}_q^N$, where $N = (q+1)^2$, is surjective, so the code is equal to $\mathbb{F}_q^{(q+1)^2}$ and the minimum distance is equal to 1.

Codes defined on rational normal scrolls

We were able to count these classes and prove that the dimension of C_d is:

(a) $(n+m)\dfrac{d(d+1)}{2} + d + 1$ if $d \leq q/n$;

(b) $(s+1)\left[\dfrac{(n+m)s}{2} + m(d-s) + 1\right] + (d-s)(q+1)$
if $m < n$ and $q/n < d \leq q/m$, where $s = \left\lfloor \dfrac{q-md}{n-m} \right\rfloor$;

(c) $(d+1)(q+1)$ if $q/m < d < q$;

(d) $(q+1)^2$ if $q \leq d$.

Observe that if $d \geq q$ then the evaluation morphism $\varphi : \mathbb{F}[\mathbf{X}]_d / I_S(d) \to \mathbb{F}_q^N$, where $N = (q+1)^2$, is surjective, so the code is equal to $\mathbb{F}_q^{(q+1)^2}$ and the minimum distance is equal to 1.

Codes defined on rational normal scrolls

We were able to count these classes and prove that the dimension of C_d is:

(a) $(n+m)\dfrac{d(d+1)}{2} + d + 1$ if $d \leq q/n$;

(b) $(s+1)\left[\dfrac{(n+m)s}{2} + m(d-s) + 1\right] + (d-s)(q+1)$
if $m < n$ and $q/n < d \leq q/m$, where $s = \left\lfloor \dfrac{q-md}{n-m} \right\rfloor$;

(c) $(d+1)(q+1)$ if $q/m < d < q$;

(d) $(q+1)^2$ if $q \leq d$.

Observe that if $d \geq q$ then the evaluation morphism $\varphi : \mathbb{F}[\mathbf{X}]_d / I_S(d) \to \mathbb{F}_q^N$, where $N = (q+1)^2$, is surjective, so the code is equal to $\mathbb{F}_q^{(q+1)^2}$ and the minimum distance is equal to 1.

Codes defined on rational normal scrolls

To estimate $d_{\min}(C_d)$, using a reasoning similar to the affine case, we proved that a lower bound for $\varphi(f + I_S(d))$ is $\#(\{M \in \Delta(\mathcal{J}) \mid \deg M = e, \operatorname{Im}(f) \mid M\})$ for $e \gg 0$. Using these results we were able to show that $d_{\min}(C_d)$ satisfies

$$\begin{aligned} (q-d+1)(q-nd+1) \leq d_{\min}(C_d) \leq q(q-nd+1) \qquad & \text{for } d \leq q/n\,, \\ q-d+1 \leq \delta_S(d) \leq q-d+1+\sigma \qquad & \text{for } q/n < d < q \end{aligned}$$

where σ is defined by

$$\sigma = \begin{cases} \left\lfloor \dfrac{q-md}{n-m} \right\rfloor & \text{if } \frac{q}{n} < d \leq \frac{q}{m} \text{ and } m < n\,; \\ \\ 0 & \text{if } \frac{q}{m} < d < q\,. \end{cases}$$

In the case when $n = m$ we manage to determine the precise values for $d_{\min}(C_d)$, which are

$$d_{\min}(C_d) = \begin{cases} (q-d+1)(q-nd+1) & \text{for } 1 \leq d \leq q/n \\ q-d+1 & \text{for } q/n \leq d \leq q \\ 1 & \text{for } q \leq d\,. \end{cases}$$

Codes defined on rational normal scrolls

To estimate $d_{\min}(C_d)$, using a reasoning similar to the affine case, we proved that a lower bound for $\varphi(f + I_S(d))$ is $\#(\{M \in \Delta(\mathcal{J}) \mid \deg M = e, \mathrm{lm}(f) \mid M\})$ for $e \gg 0$. Using these results we were able to show that $d_{\min}(C_d)$ satisfies

$$\begin{array}{ll} (q-d+1)(q-nd+1) \leq d_{\min}(C_d) \leq q(q-nd+1) & \text{for } d \leq q/n\,, \\ q-d+1 \leq \delta_S(d) \leq q-d+1+\sigma & \text{for } q/n < d < q \end{array}$$

where σ is defined by

$$\sigma = \begin{cases} \left\lfloor \dfrac{q-md}{n-m} \right\rfloor & \text{if } \frac{q}{n} < d \leq \frac{q}{m} \text{ and } m < n\,; \\ 0 & \text{if } \frac{q}{m} < d < q\,. \end{cases}$$

In the case when $n = m$ we manage to determine the precise values for $d_{\min}(C_d)$, which are

$$d_{\min}(C_d) = \begin{cases} (q-d+1)(q-nd+1) & \text{for } 1 \leq d \leq q/n \\ q-d+1 & \text{for } q/n \leq d \leq q \\ 1 & \text{for } q \leq d\,. \end{cases}$$

Codes defined on rational normal scrolls

To estimate $d_{\min}(C_d)$, using a reasoning similar to the affine case, we proved that a lower bound for $\varphi(f + I_S(d))$ is $\#(\{M \in \Delta(\mathcal{J}) \mid \deg M = e, \operatorname{lm}(f) \mid M\})$ for $e \gg 0$. Using these results we were able to show that $d_{\min}(C_d)$ satisfies

$$\begin{array}{ll} (q-d+1)(q-nd+1) \leq d_{\min}(C_d) \leq q(q-nd+1) & \text{for } d \leq q/n\,, \\ q-d+1 \leq \delta_S(d) \leq q-d+1+\sigma & \text{for } q/n < d < q \end{array}$$

where σ is defined by

$$\sigma = \begin{cases} \left\lfloor \dfrac{q-md}{n-m} \right\rfloor & \text{if } \frac{q}{n} < d \leq \frac{q}{m} \text{ and } m < n\,; \\ 0 & \text{if } \frac{q}{m} < d < q\,. \end{cases}$$

In the case when $n = m$ we manage to determine the precise values for $d_{\min}(C_d)$, which are

$$d_{\min}(C_d) = \begin{cases} (q-d+1)(q-nd+1) & \text{for } 1 \leq d \leq q/n \\ q-d+1 & \text{for } q/n \leq d \leq q \\ 1 & \text{for } q \leq d\,. \end{cases}$$

Codes defined on rational normal scrolls

To estimate $d_{\min}(C_d)$, using a reasoning similar to the affine case, we proved that a lower bound for $\varphi(f + I_S(d))$ is $\#(\{M \in \Delta(\mathcal{J}) \mid \deg M = e, \mathrm{lm}(f) \mid M\})$ for $e \gg 0$. Using these results we were able to show that $d_{\min}(C_d)$ satisfies

$$\begin{aligned} (q-d+1)(q-nd+1) \le d_{\min}(C_d) \le q(q-nd+1) \qquad & \text{for } d \le q/n\,, \\ q-d+1 \le \delta_S(d) \le q-d+1+\sigma \qquad & \text{for } q/n < d < q \end{aligned}$$

where σ is defined by

$$\sigma = \begin{cases} \left\lfloor \dfrac{q-md}{n-m} \right\rfloor & \text{if } \frac{q}{n} < d \le \frac{q}{m} \text{ and } m < n\,; \\ 0 & \text{if } \frac{q}{m} < d < q\,. \end{cases}$$

In the case when $n = m$ we manage to determine the precise values for $d_{\min}(C_d)$, which are

$$d_{\min}(C_d) = \begin{cases} (q-d+1)(q-nd+1) & \text{for } 1 \le d \le q/n \\ q-d+1 & \text{for } q/n \le d \le q \\ 1 & \text{for } q \le d\,. \end{cases}$$

Codes defined on rational normal scrolls

To estimate $d_{\min}(C_d)$, using a reasoning similar to the affine case, we proved that a lower bound for $\varphi(f + I_S(d))$ is $\#(\{M \in \Delta(\mathcal{J}) \mid \deg M = e, \mathrm{Im}(f) \mid M\})$ for $e \gg 0$. Using these results we were able to show that $d_{\min}(C_d)$ satisfies

$$(q-d+1)(q-nd+1) \le d_{\min}(C_d) \le q(q-nd+1) \qquad \text{for } d \le q/n\,,$$
$$q-d+1 \le \delta_S(d) \le q-d+1+\sigma \qquad \text{for } q/n < d < q$$

where σ is defined by

$$\sigma = \begin{cases} \left\lfloor \dfrac{q-md}{n-m} \right\rfloor & \text{if } \frac{q}{n} < d \le \frac{q}{m} \text{ and } m < n\,; \\ 0 & \text{if } \frac{q}{m} < d < q\,. \end{cases}$$

In the case when $n = m$ we manage to determine the precise values for $d_{\min}(C_d)$, which are

$$d_{\min}(C_d) = \begin{cases} (q-d+1)(q-nd+1) & \text{for } 1 \le d \le q/n \\ q-d+1 & \text{for } q/n \le d \le q \\ 1 & \text{for } q \le d\,. \end{cases}$$

Codes defined on rational normal scrolls

To estimate $d_{\min}(C_d)$, using a reasoning similar to the affine case, we proved that a lower bound for $\varphi(f + I_S(d))$ is $\#(\{M \in \Delta(\mathcal{J}) \mid \deg M = e, \operatorname{lm}(f) \mid M\})$ for $e \gg 0$. Using these results we were able to show that $d_{\min}(C_d)$ satisfies

$$\begin{array}{lr} (q-d+1)(q-nd+1) \leq d_{\min}(C_d) \leq q(q-nd+1) & \text{for } d \leq q/n\,, \\ q-d+1 \leq \delta_S(d) \leq q-d+1+\sigma & \text{for } q/n < d < q \end{array}$$

where σ is defined by

$$\sigma = \begin{cases} \left\lfloor \dfrac{q-md}{n-m} \right\rfloor & \text{if } \frac{q}{n} < d \leq \frac{q}{m} \text{ and } m < n\,; \\ 0 & \text{if } \frac{q}{m} < d < q\,. \end{cases}$$

In the case when $n = m$ we manage to determine the precise values for $d_{\min}(C_d)$, which are

$$d_{\min}(C_d) = \begin{cases} (q-d+1)(q-nd+1) & \text{for } 1 \leq d \leq q/n \\ q-d+1 & \text{for } q/n \leq d \leq q \\ 1 & \text{for } q \leq d\,. \end{cases}$$

Codes defined on rational normal scrolls

To estimate $d_{\min}(C_d)$, using a reasoning similar to the affine case, we proved that a lower bound for $\varphi(f + I_S(d))$ is $\#(\{M \in \Delta(\mathcal{J}) \mid \deg M = e, \operatorname{Im}(f) \mid M\})$ for $e \gg 0$. Using these results we were able to show that $d_{\min}(C_d)$ satisfies

$$\begin{aligned} (q-d+1)(q-nd+1) \leq d_{\min}(C_d) \leq q(q-nd+1) \qquad & \text{for } d \leq q/n\,, \\ q-d+1 \leq \delta_S(d) \leq q-d+1+\sigma \qquad & \text{for } q/n < d < q \end{aligned}$$

where σ is defined by

$$\sigma = \begin{cases} \left\lfloor \dfrac{q-md}{n-m} \right\rfloor & \text{if } \frac{q}{n} < d \leq \frac{q}{m} \text{ and } m < n\,; \\ 0 & \text{if } \frac{q}{m} < d < q\,. \end{cases}$$

In the case when $n = m$ we manage to determine the precise values for $d_{\min}(C_d)$, which are

$$d_{\min}(C_d) = \begin{cases} (q-d+1)(q-nd+1) & \text{for } 1 \leq d \leq q/n \\ q-d+1 & \text{for } q/n \leq d \leq q \\ 1 & \text{for } q \leq d\,. \end{cases}$$

THANK YOU!