Group Symmetries of Complementary Code Matrices

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This work was done in collaboration with Professor Hieu D. Nguyen at Rowan University.

And the topic was inspired by Greg Coxson

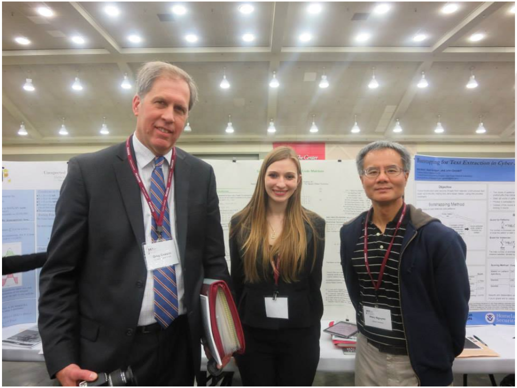

Figure: JMM 2014 Undergraduate Poster Session

Barker Codes and Golay Pairs

Barker Codes and Golay Pairs

- When a radar signal is sent out the returned signal is compared to it by computing the aperiodic auto correlation function, A, as follows:

$$\mathrm{A}_x(j) = \begin{cases} \sum_{i=1}^{N-j} a_i \bar{a}_{i+j}, & \text{if } 0 \leq j \leq N-1; \\ \overline{\mathrm{A}_x(-j)}, & \text{if } -N+1 \leq j < 0. \end{cases} \quad (1)$$

Barker Codes

Definition

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Note: The even case has been proven up to $N = 4 \times 1980483001226429873804 1^2$

Example of a Binary Code

The autocorrelation function for a single code, $x = \{1, 1, 1, -1\}$ of length $N = 4$, can be computed as follows:

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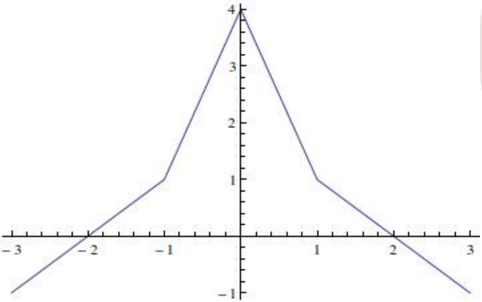
- $A_x(2) = \bar{A}_x(-2) = \sum_{i=1}^{N-2} x_i x_{i+2}^c = (1 \cdot 1) + (1 \cdot -1) = 0$
- $A_x(3) = \bar{A}_x(-3) = \sum_{i=1}^{N-3} x_i x_{i+3}^c = (1 \cdot -1) = -1$

Example of a Binary Code

$$x = \{1, 1, 1, -1\} \text{ with } A_x = \{-1, 0, 1, 4, 1, 0, -1\}$$



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$$y = \{1, 1, -1, 1\} \text{ with } A_y = \{1, 0, -1, 4, -1, 0, 1\}$$

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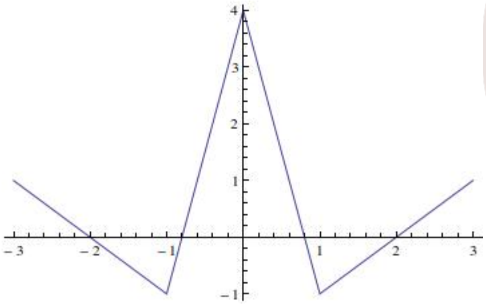

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- $A_x + A_y$
- $\{-1, 0, 1, 4, 1, 0, -1\} + \{1, 0, -1, 4, -1, 0, 1\}$

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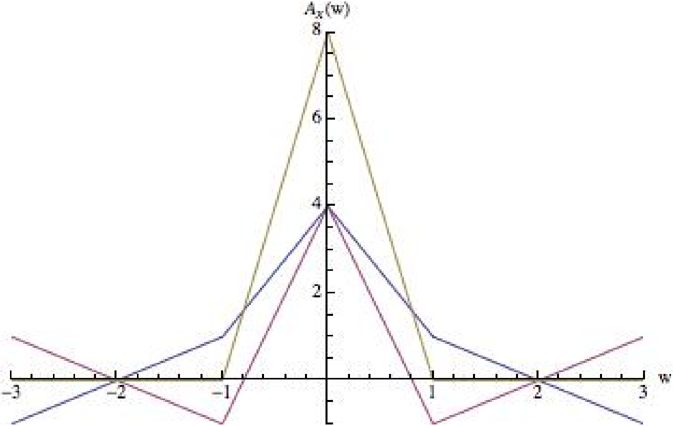
Composite autocorrelation for pair of codes $x = \{1, 1, 1, -1\}$ and $y = \{1, 1, -1, 1\}$:

- $A_x + A_y$
- $\{-1, 0, 1, 4, 1, 0, -1\} + \{1, 0, -1, 4, -1, 0, 1\}$

$$\{0, 0, 0, 8, 0, 0, 0\}$$

This is called a Golay Pair.

Graph of A_x, A_y and $A_x + A_y$



Complementary Code Matrices

Definition

A **complementary code matrix** (or CCM) is a $N \times K$ matrix M whose row Gramian, $B = M \cdot M^*$, is diagonally regular with diagonal entries equal to K. Here, * represents the conjugate transpose.

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$$p\text{-phase: } \exp\left(\frac{2\pi i}{p} + \frac{2k\pi i}{p}\right) \text{ with } k = 0, \ldots, p-1$$

We define the set of $N \times K$ p-phase CCMs as $C_{N,K}(p)$.

Example

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & i & -i \\ -1 & -1 & i & -i \\ 1 & 1 & -1 & -1 \end{pmatrix}$$

Example

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & i & -i \\ -1 & -1 & i & -i \\ 1 & 1 & -1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & -i & -i & -1 \\ 1 & i & i & -1 \end{pmatrix} =$$

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$$\begin{pmatrix} 4 & 2 & -2 & 0 \\ 2 & 4 & 0 & 2 \\ -2 & 0 & 4 & -2 \\ 0 & 2 & -2 & 4 \end{pmatrix}$$

Our Research

- Described relations between symmetries of Complmentary Code Matrices
- Adapted an existing algorithm to speed up search for CCMs
- Classified the matrices into different equivalence classes
- Created a new construction method

Symmetries

Coxson-Haloupek [1]

Suppose that M is an $N \times K$ CCM. Then we can obtain an equivalent $N \times K$ CCM using the following transformations.

(i) Column multiplication by a unimodular complex number.
(ii) Column conjugate reversal.
(iii) Matrix conjugation.
(iv) Progressive multiplication by consecutive powers of a unimodular complex number.
(v) Column permutation.

Example

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ i & -i & i & -i \\ i & -i & -i & i \end{pmatrix} \xrightarrow{x_1 \times i}$$

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Example

$$\begin{pmatrix} i & 1 & 1 & 1 \\ i & 1 & -1 & -1 \\ -1 & -i & i & -i \\ -1 & -i & -i & i \end{pmatrix} \xrightarrow{rev(\bar{x}_2)}$$

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$$\begin{pmatrix} -i & -i & 1 & 1 \\ -i & -i & -1 & -1 \\ -1 & 1 & -i & i \\ -1 & 1 & i & -i \end{pmatrix} \xrightarrow{Q(i)}$$

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$$\begin{pmatrix} 1 & 1 & i & i \\ i & i & 1 & 1 \\ i & -i & -1 & 1 \\ -1 & 1 & i & -i \end{pmatrix} \xrightarrow{(x_1,x_3)(x_2,x_4)}$$

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Quick Proof

$$U = \{\alpha_1, \alpha_2, \ldots, \alpha_K\}$$

$$T = \{t_1, t_2, \ldots, t_K\} \text{ where } t = 0, 1$$

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$$C_U \rho_T M = C_U \rho_T [x_1, x_2, \ldots, x_K]$$

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$$\begin{aligned} C_U \rho_T M &= C_U \rho_T [x_1, x_2, \ldots, x_K] \\ &= C_U[\omega^{2-t_1}(x_1), \omega^{2-t_2}(x_2), \ldots, \omega^{2-t_K}(x_K)] \end{aligned}$$

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&= [\alpha_1 \omega^{2-t_1}(x_1), \alpha_2 \omega^{2-t_2}(x_2), \ldots, \alpha_K \omega^{2-t_K}(x_K)] \\
&= \rho_T [\omega^{2-t1}(\alpha_1) x_1, \omega^{2-t_2}(\alpha_2) x_2, \ldots, \omega^{2-t_K}(\alpha_K) x_K] \\
&= \rho_T C_{U_T} [x_1, x_2, x_3, \ldots, x_K] \\
&= \rho_T C_{U_T} M
\end{aligned}$$

Definition

The **complementary group**, G, of the set of all $N \times K$ p-phase CCMs is defined to be the group generated by the symmetries $S, P, C_U, \rho_T, Q(\beta)$ and their relations given in the following lemma.

Lemma

Let M be a $N \times K$ p-phase CCM, $M = [x_1, x_2, x_3, \ldots, x_K]$ and $x_k = [m_{1,k}, m_{2,k}, m_{3,k}, \ldots, m_{N,k}]^T$. Then

(i) $C_U \rho_T M = \rho_T C_{U_T} M$

(ii) $C_U SM = SC_{\bar{U}} M$

(iii) $C_U Q(\beta)M = Q(\beta)C_U M$

(iv) $C_U PM = PC_{U_P} M$

(v) $\rho_T SM = S\rho_T M$

(vi) $\rho_T PM = P\rho_{T_{P^{-1}}} M$

(vii) $SQ(\beta)M = Q(\bar{\beta})SM$

(viii) $SPM = PSM$

(ix) $Q(\beta)\rho_T M = C_{U_{T,\beta}} \rho_T Q_\beta M$

(x) $Q(\beta)PM = PQ(\beta)M$

Theorem

The cardinality of the complementary group G of $C_{N,K}(p)$ is bounded by

$$|G| \leq 2^{K+1} p^{K+1} K! \quad (2)$$

Proof

Each CCM preserving operation on the matrix M can be represented in the following form.

$$SPC_u \rho_T Q(\beta)$$

$|S| = 2$
$|P| = K!$
$|C_U| = p^K$
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$|Q(\beta)| = p$

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So this will will produce a max of

$$|G| \leq |S||P||C_U||\rho_T||Q(\beta)| = 2K!p^K 2^K p = 2^{K+1}p^{K+1}K!$$

CCMs from M.

Applications of the Symmetries

$$M = \begin{pmatrix} m_{1,1} & m_{1,2} & m_{1,3} & m_{1,4} \\ m_{2,1} & m_{2,2} & m_{2,3} & m_{2,4} \\ m_{3,1} & m_{3,2} & m_{3,3} & m_{3,4} \\ m_{4,1} & m_{4,2} & m_{4,3} & m_{4,4} \end{pmatrix}$$

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$$U_1 = (\bar{m}_{1,1}, \bar{m}_{1,2}, \bar{m}_{1,3}, \bar{m}_{1,4})$$

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$$U_2 = (m_{2,1}\bar{m}_{1,1}, m_{2,1}\bar{m}_{1,1}, m_{2,1}\bar{m}_{1,1}, m_{2,1}\bar{m}_{1,1})$$

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$$\beta = \bar{m}_{2,1} m_{1,1}$$

$$Q(\beta) C_{U_2} C_{U_1} M$$

Applications of the Symmetries

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & m_{2,2} & m_{2,3} & m_{2,4} \\ m_{3,1} & m_{3,2} & m_{3,3} & m_{3,4} \\ m_{4,1} & m_{4,2} & m_{4,3} & m_{4,4} \end{pmatrix}$$

$$4^{16} \to 4^{11}$$

Applications of the Symmetries

$$\begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{pmatrix}$$

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$$\begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{pmatrix} \cdot \begin{pmatrix} \bar{r}_1 & \bar{r}_2 & \bar{r}_3 & \bar{r}_4 \end{pmatrix}$$

Applications of the Symmetries

$$\begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{pmatrix} \cdot \begin{pmatrix} \bar{r}_1 & \bar{r}_2 & \bar{r}_3 & \bar{r}_4 \end{pmatrix} = \begin{pmatrix} r_1\bar{r}_1 & r_1\bar{r}_2 & r_1\bar{r}_3 & r_1\bar{r}_4 \\ r_2\bar{r}_1 & r_2\bar{r}_1 & r_2\bar{r}_1 & r_2\bar{r}_1 \\ r_3\bar{r}_1 & r_3\bar{r}_2 & r_3\bar{r}_3 & r_3\bar{r}_4 \\ r_4\bar{r}_1 & r_4\bar{r}_2 & r_4\bar{r}_3 & r_4\bar{r}_4 \end{pmatrix}$$

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$$r_4\bar{r}_1 = 0$$

Equivalence Classes

$N \times K$ 4-phase CCMs	Coxson-Russo Algorithm	Total Number of Equivalence Classes	Hadamard Representations
2x4	36	2	2
3x4	95	5	5
4x4	231	24	17
5x4	5246	133	0
6x4	23448	1448	0

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Exhaustive 4×4 4-phase CCM $\approx$ 4.3 million

4×4 Equivalence Class Representations

1. $[[1,1,1,1],[1,1,1,1],[1,1,-1,-1],[-1,-1,1,1]]$
2. $[[1,1,1,1],[1,1,1,i],[-1,-1,-i,1],[1,-1,1,-1]]$
3. $[[1,1,1,1],[1,1,i,i],[1,-1,i,-i],[-1,1,1,-1]]$
4. $[[1,1,1,1],[1,1,i,i],[i,-i,1,-1],[1,-1,-1,1]]$
5. $[[1,1,1,1],[1,1,-1,-1],[1,-1,1,-1],[1,-1,-1,1]]$
6. $[[1,1,1,1],[1,i,i,-1],[-1,i,i,1],[-1,1,1,-1]]$
7. $[[1,1,1,1],[1,1,1,1],[1,i,-1,-i],[-1,-i,1,i]]$
8. $[[1,1,1,1],[1,1,i,i],[1,-1,1,-1],[-1,1,-i,i]]$
9. $[[1,1,1,1],[1,1,i,i],[i,-1,1,-i],[1,-1,-i,i]]$
10. $[[1,1,1,1],[1,1,i,i],[-1,-1,1,1],[1,-1,i,-i]]$
11. $[[1,1,1,1],[1,1,i,i],[i,-i,i,-i],[1,-1,-i,i]]$
12. $[[1,1,1,1],[1,1,-1,-1],[1,-1,i,-i],[1,-1,-i,i]]$

4×4 Equivalence Class Representations Continued

13. $[[1,1,1,1],[1,1,i,-1],[1,-i,-1,1],[i,-1,1,-i]]$
14. $[[1,1,1,1],[1,1,i,-1],[1,-i,-i,i],[i,-1,1,-i]]$
15. $[[1,1,1,1],[1,1,1,i],[i,-1,-i,-i],[-i,i,1,-1]]$
16. $[[1,1,1,1],[1,1,1,i],[-1,-1,-i,1],[i,-i,1,-1]]$
17. $[[1,1,1,1],[1,1,i,-1],[i,-1,1,-1],[1,-i,-1,i]]$
18. $[[1,1,1,1],[1,1,i,i],[i,-i,i,-i],[-i,i,1,-1]]$
19. $[[1,1,1,1],[1,1,1,1],[i,i,-i,-i],[-i,-i,i,i]]$
20. $[[1,1,1,1],[1,1,-1,-1],[i,-i,i,-i],[i,-i,-i,i]]$
21. $[[1,1,1,1],[1,i,i,-1],[-i,-1,-1,i],[-i,i,i,-i]]$
22. $[[1,1,1,1],[1,i,-1,-i],[-i,i,-i,i],[-i,-1,i,1]]$
23. $[[1,1,1,1],[1,i,-1,-i],[-1,1,-1,1],[-1,i,1,-i]]$
24. $[[1,1,1,1],[1,i,-1,-i],[1,-1,1,-1],[1,-i,-1,i]]$

Kronecker Product

Let A and B be 2×2 p-phase CCMs.

$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \oplus \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix} =$$

$$\begin{pmatrix} a_{1,1}b_{1,1} & a_{1,1}b_{1,2} & a_{1,2}b_{1,1} & a_{1,2}b1,2 \\ a_{1,1}b_{2,1} & a_{1,1}b_{2,2} & a_{1,2}b_{2,1} & a_{1,2}b_{2,2} \\ a_{2,1}b_{1,1} & a_{2,1}b_{1,2} & a_{2,2}b_{1,1} & a_{2,2}b_{1}1,2 \\ a_{2,1}b_{2,1} & a_{2,1}b_{2,2} & a_{2,2}b_{2,1} & a_{2,2}b_{2,2} \end{pmatrix}$$

Concatentation Theorem

Let A and B be 4×2 p-phase CCMs.

$$\left[\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \\ a_{4,1} & a_{4,2} \end{pmatrix}, \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \\ b_{3,1} & b_{3,2} \\ b_{4,1} & b_{4,2} \end{pmatrix}\right] = \begin{pmatrix} a_{1,1} & a_{1,2} & b_{1,1} & b_{1,2} \\ a_{2,1} & a_{2,2} & b_{2,1} & b_{2,2} \\ a_{3,1} & a_{3,2} & b_{3,1} & b_{3,2} \\ a_{4,1} & a_{4,2} & b_{4,1} & b_{4,2} \end{pmatrix}$$

Dual Pair Theorem

$$M = \begin{pmatrix} -1 & -1 & -i & -i \\ -i & -i & i & i \\ i & -i & -1 & 1 \\ i & -i & i & -i \end{pmatrix}$$

Dual Pair Theorem

$$M = \begin{pmatrix} -1 & -1 & -i & -i \\ -i & -i & i & i \\ i & -i & -1 & 1 \\ i & -i & i & -i \end{pmatrix}$$

$$M = A + iB$$

Dual Pair Theorem

$$M = \begin{pmatrix} -1 & -1 & -i & -i \\ -i & -i & i & i \\ i & -i & -1 & 1 \\ i & -i & i & -i \end{pmatrix}$$

$$M = A + iB$$

$$A = \begin{pmatrix} -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & -1 & 1 & -1 \end{pmatrix}$$

Dual Pair Theorem

Assume that A and B are $N \times K$ ternary, $\{-1, 0, 1\}$ "CCMs".
Then $Z = A + iB$ is a $N \times K$ quad-phase CCM if
(i) $\mid A_{n,k} \mid + \mid B_{n,k} \mid = 1 \; \{\forall n, k | 1 \leq n \leq N \text{ and } 1 \leq k \leq K\}$
(ii) $BA^* - B^*A$ is diagonally regular

Proof

Prove Z is quad-phase.

Prove ZZ^* is Diagonally Regular.

Proof

Prove Z is quad-phase.
Assumed $| A_{n,k} | + | B_{n,k} | = 1$ which implies $-1, 1, i, -i$.

Prove ZZ^* is Diagonally Regular.

Proof

Prove Z is quad-phase.
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Prove ZZ^* is Diagonally Regular.
Z = A+iB so the following holds true.

$$\begin{aligned} ZZ^* &= (A + iB)(A + iB)^* \\ &= (A + iB)(A^* - iB^*) \\ &= AA^* + i(BA^* - B^*A) + BB^* \end{aligned}$$

Proof

Prove Z is quad-phase.
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QED!

Constructing the Equivalency Classes

4-phase CCMs	Equivalence Classes	Kronecker Product	Concatentation Theorem	CCM Dual Pair
2x4	2	n/a	2	2
3x4	5	n/a	1	5
4x4	24	2	6	22
5x4	133	n/a	3	94
6x4	1448	2	27	471

For More Information...

- CCM Website: elvis.rowan.edu/datamining/ccm/
- Equivlency Class Links: elvis.rowan.edu/datamining/ccm/equivalence/
- Our Paper: arxiv.org/abs/1506.00011
- My email: brookelogan974@gmail.com

Thank You!

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