Algebraically defined graphs and generalized quadrangles

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Cages and the Moore bound

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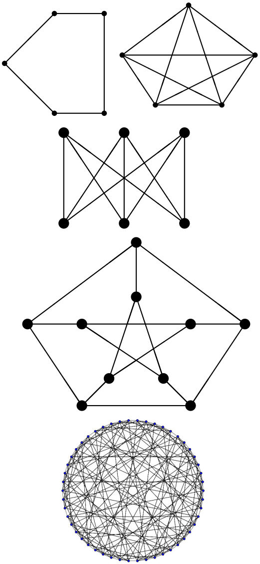
Key idea: It is not always possible to construct a graph that meets the Moore bound.

When can the Moore Bound (potentially) be achieved?

The ONLY possible parameters and examples:

k	g	Unique (k,g)-cage meeting the lower bound
2	g	C_g
k	3	K_{k+1}
k	4	$K_{k,k}$
3	5	Petersen graph
7	5	Hoffman-Singleton graph
57	5	?????
k	6	Incidence graphs of generalized 3-gons of prime power order $k-1$
k	8	Incidence graphs of generalized 4-gons of prime power order $k-1$
k	12	Incidence graphs of generalized 6-gons of prime power order $k-1$



Wikipedia(Uzyel)

Brief intermission: What about a $(57, 5)$-cage?

- A $(57, 5)$-cage would have 3250 vertices and diameter 2.
- The eigenvalues of such a graph's adjacency matrix 57, 7, and -8 (with multiplicities 1, 1729, and 1520, respectively).
- Properties of the automorphism group of a $(57, 5)$-cage have been studied.

What is a Generalized Quadrangle?

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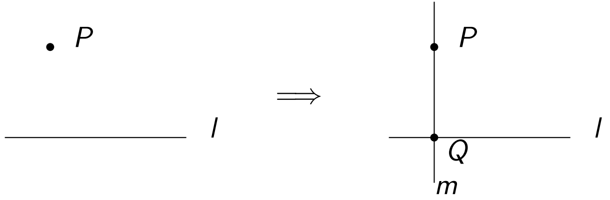
What is a Generalized Quadrangle?

Definition

A **generalized quadrangle of order** q is an incidence structure of $q^3 + q^2 + q + 1$ points and $q^3 + q^2 + q + 1$ lines such that...

1. Every point lies on $q + 1$ lines; two distinct points determine at most one line.
2. Every line contains $q + 1$ points; two distinct lines have at most one point in common.
3. If P is a point and L is a line such that P is not on L, then there exists a unique line that contains P and intersects L.



An example: $GQ(1)$

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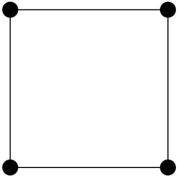

$GQ(q)$: An alternate characterization

Definition

A **generalized quadrangle of order** q is an incidence structure whose connected (bipartite) point-line incidence graph:

1. is $(q+1)$-regular (every vertex is connected to $q+1$ others)
2. has girth eight (there are no cycles of length less than eight)
3. has diameter four (the distance between any two vertices is at most four)

Example: An alternate characterization of $GQ(1)$

The point-line incidence graph of $GQ(1)$...

1. is two-regular
2. has girth eight
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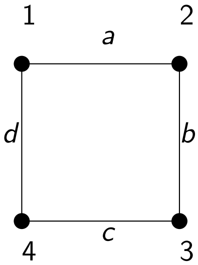

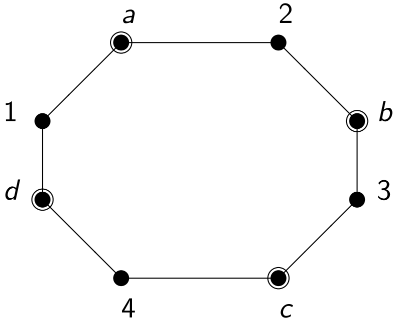

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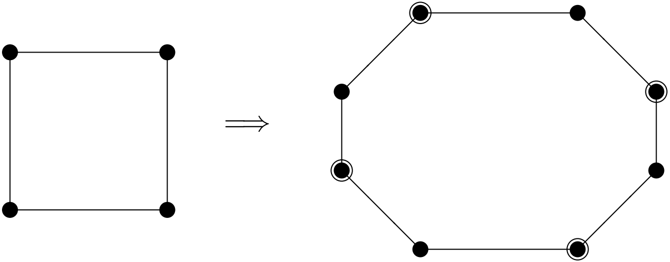
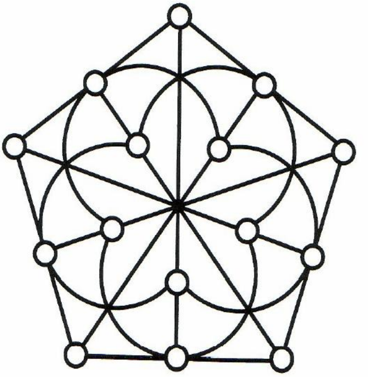


Example: $GQ(2)$



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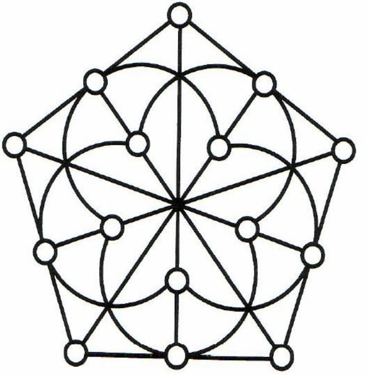
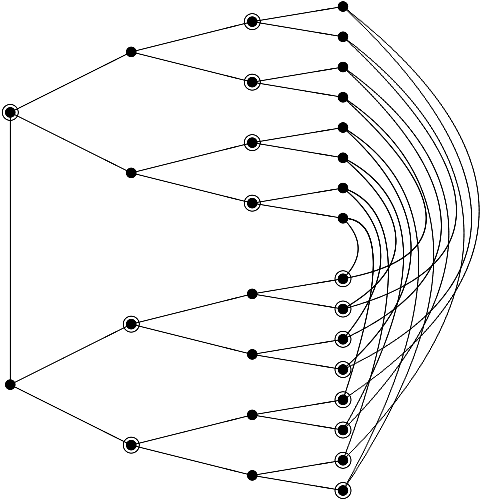


$\Longrightarrow$



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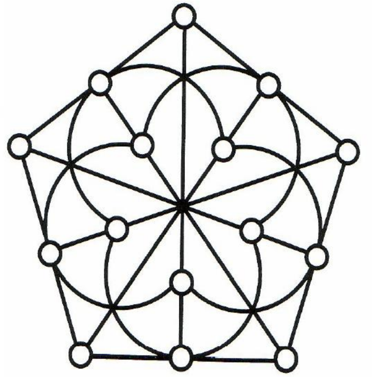

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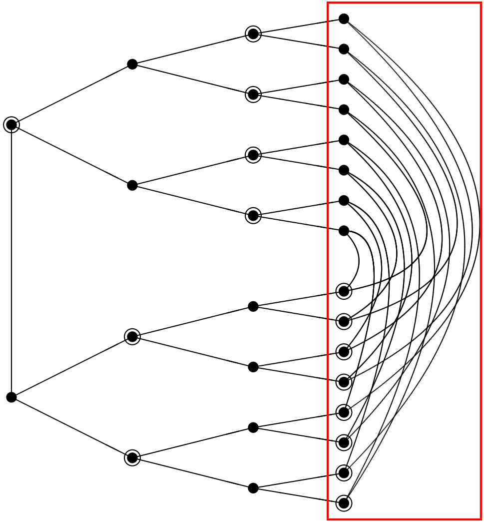
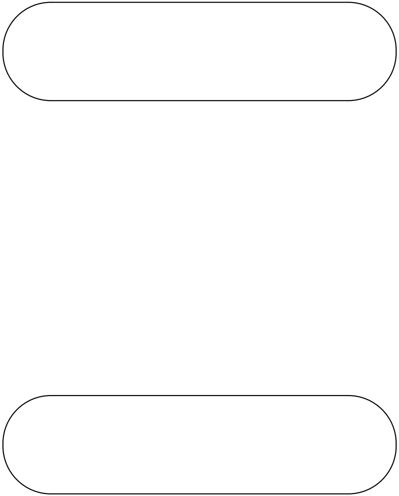


How can we represent this boxed subgraph?

We will construct a family of bipartite graphs as follows.

Let $\mathbb{F}$ be a field.

Let $f(x, y)$ and $g(x, y)$ be bivariate polynomials with coefficients in $\mathbb{F}$.



Connection to Algebraically Defined Graphs

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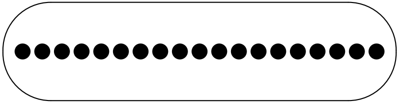
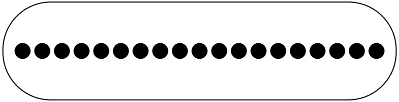


$P = \mathbb{F}^3 = \{(x_1, x_2, x_3) | x_i \in \mathbb{F}\}$



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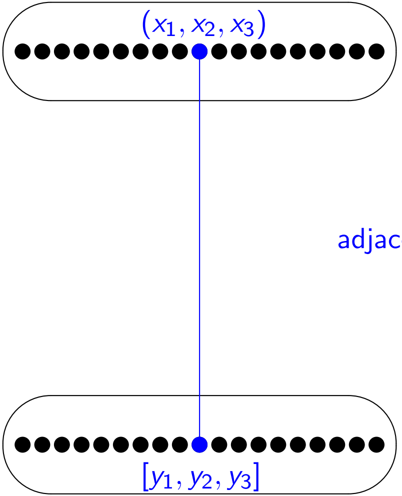

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$$\text{adjacency iff } \begin{cases} x_2 + y_2 = f(x_1, y_1) \\ x_3 + y_3 = g(x_1, y_1) \end{cases}$$

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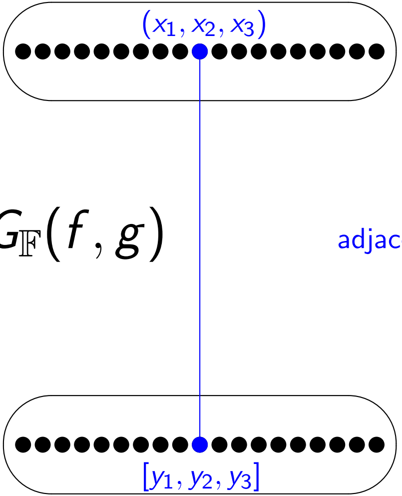

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$G_{\mathbb{F}}(f, g)$

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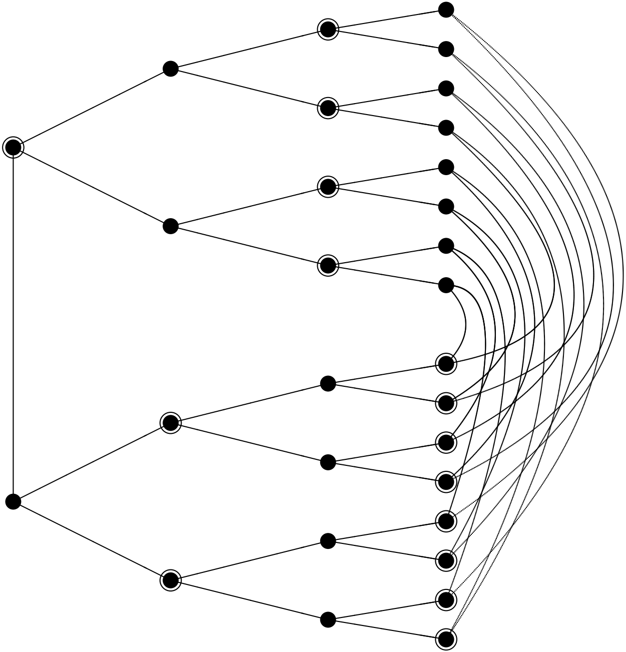
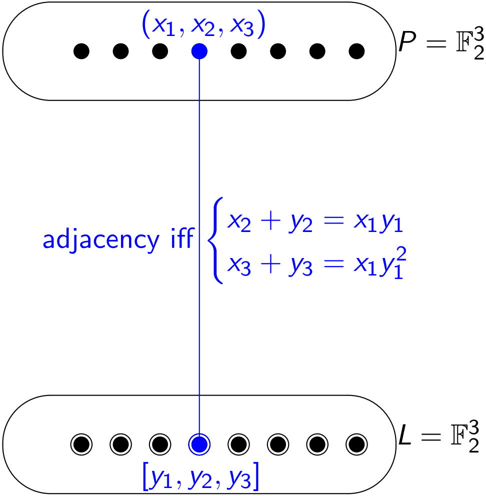


$q = 2$

$G_{\mathbb{F}_2}(xy, xy^2)$



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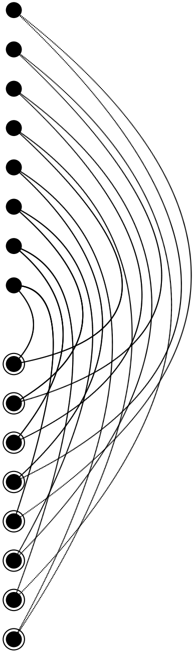

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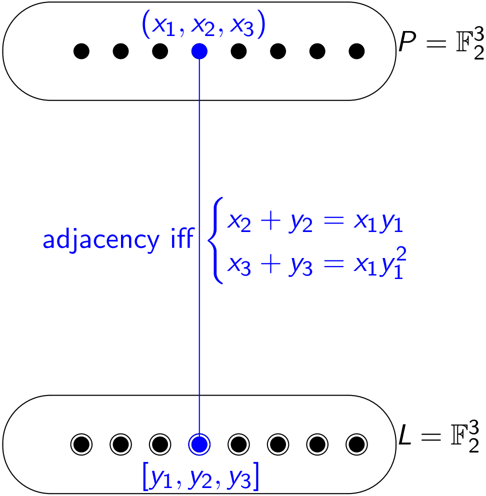
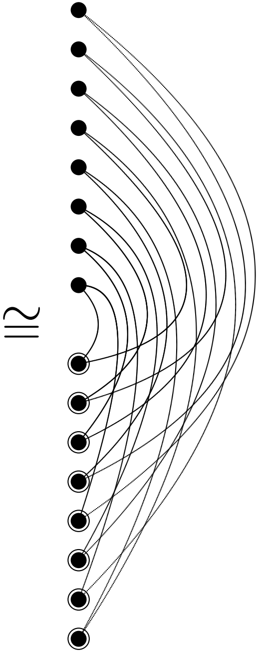
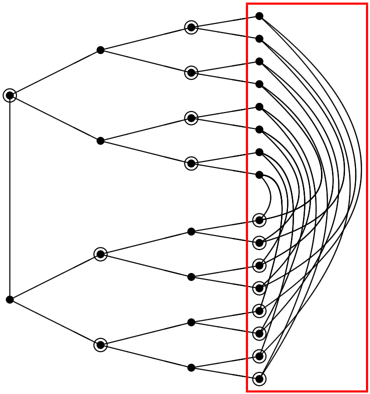




Generalizes to any $\mathbb{F}_q$

Question: Is $G_{\mathbb{F}}(xy, xy^2)$ the unique girth eight algebraically defined graph (up to isomorphism)?



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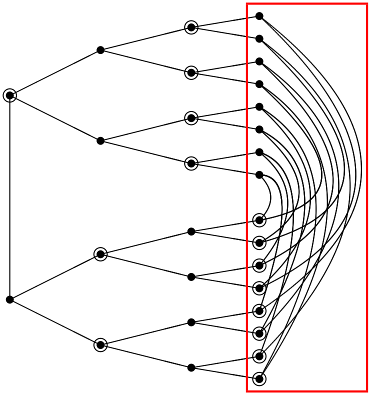

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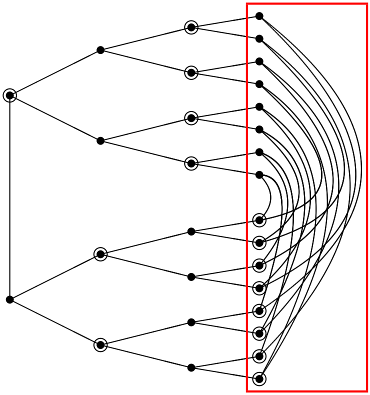
- If so, we have an interesting characterization.
- If not, then we might be able to construct a new generalized quadrangle by replacing the boxed subgraph with this new girth eight graph. This is interesting because for a given odd prime power, only one GQ is known (up to isomorphism). Also, in the even order case, additional GQs can be constructed in this way.



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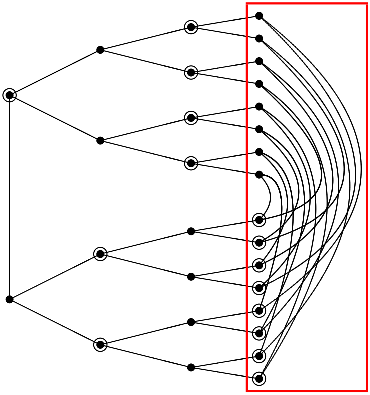

Conjecture (V. Dmytrenko, F. Lazebnik, J. Williford; 2007)

$G_{\mathbb{F}}(xy, xy^2)$ is the unique girth eight algebraically defined graph (up to isomorphism)

Connection to permutation polynomials

Theorem (V. Dmytrenko, F. Lazebnik, J. Williford; 2007)

Let $q = p^e$ be an odd prime power. Then every monomial graph of girth at least eight is isomorphic to the graph $G_q(xy, x^k y^{2k})$, where k is not divisible by p. If $q \geq 5$, then:

1. $((x+1)^{2k} - 1)x^{q-1-k} - 2x^{q-1} \in \mathbb{F}_q[x]$ *is a permutation polynomial of* $\mathbb{F}_q$.
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Theorem (Hermite-Dickson criterion)

Let $\mathbb{F}_q$ be of characteristic p. Then $f \in \mathbb{F}_q[x]$ is a permutation polynomial of $\mathbb{F}_q$ if and only if the following two conditions hold:

1. f *has exactly one root in* $\mathbb{F}_q$.
2. *for each integer t with $1 \leq t \leq q-2$ and $p \nmid t$, the reduction of f^t* $\pmod{x^q - x}$ *has degree at most $q-2$.*

A Key Result

Theorem (V. Dmytrenko, F. Lazebnik, J. Williford; 2007)

Let $q = p^e$ be an odd prime power, with $e = 2^a 3^b$ for integers $a, b \geq 0$ and $p \geq 5$.
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- What about q of other forms?

Additional Results

Theorem (BGK, 2012)

Let $q = p^e$ be an odd prime power, with $p \geq p_0$, a lower bound that depends only on the largest prime divisor of e.
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Example (What is p_0?)

- If $e = 2^a 3^b 5^c$ for integers $a, b, c \geq 0$, then $p \geq p_0 = 7$
- If $e = 2^a 3^b 5^c 7^d$ for integers $a, b, c, d \geq 0$, then $p \geq p_0 = 11$.
- If $e = 2^a 3^b 5^c 7^d 11^y$ for integers $a, b, c, d, y \geq 0$, then $p \geq p_0 = 13$.

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This implies that for any q of this form, $G_q(xy, xy^2)$ is the unique girth eight monomial graph (up to isomorphism).

Breaking News!

Theorem (Hou, Lappano, Lazebnik, posted on ArXiv a few days ago)

Let q be an odd prime power. Then every girth eight monomial graph $G_q(x^u y^v, x^k y^m)$ is isomorphic to $G_q(xy, xy^2)$.

Can we say more over a different field?

Theorem (F. Lazebnik, J. Williford, and BGK)

Every girth eight polynomial graph $G_{\mathbb{C}}(x^k y^m, f)$, where $f \in \mathbb{C}[x, y]$, is isomorphic to $G_{\mathbb{C}}(xy, xy^2)$.

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Every girth eight polynomial graph $G_{\mathbb{C}}(x^k y^m, f)$*, where* $f \in \mathbb{C}[x, y]$*, is isomorphic to* $G_{\mathbb{C}}(xy, xy^2)$*.*

Note that f in the above theorem does not need to be a monomial; the result holds for any polynomial.

Connection back to the finite field case

Theorem (Lefschetz Principle)

Let ϕ be a sentence in the language of rings. The following are equivalent.

1. ϕ *is true in the complex numbers.*
2. ϕ *is true in every algebraically closed field of characteristic zero.*
3. ϕ *is true in some algebraically closed field of characteristic zero.*
4. *There are arbitrarily large primes p such that ϕ is true in some algebraically closed field of characteristic p.*
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Informally, if $G_{\mathbb{C}}(x^k y^m, f)$ is not a candidate to replace $G_{\mathbb{C}}(xy, xy^2)$, then $G_q(x^k y^m, \hat{f})$ is not a candidate to replace $G_q(xy, xy^2)$ for "many" q.

Future Work

Open Problem

Do there exist f *and* g *in* $\mathbb{C}[x, y]$ *such that* $G = G_{\mathbb{C}}(f, g)$ *has girth eight and* G *is not isomorphic to* $G_{\mathbb{C}}(xy, xy^2)$*?*

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1. This is done when g is a monomial (i.e. when $g = \alpha x^k y^m$ for $\alpha \in \mathbb{C}$).

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1. This is done when g is a monomial (i.e. when $g = \alpha x^k y^m$ for $\alpha \in \mathbb{C}$).
2. G has a 4-cycle if and only if there exists a solution a, b, x, y to the system
$$\begin{cases} f(a,x) - f(b,x) + f(b,y) - f(a,y) = 0 \\ g(a,x) - g(b,x) + g(b,y) - g(a,y) = 0 \\ a \neq b, \;\; x \neq y. \end{cases}$$

Future Work

Open Problem

Do there exist f and g in $\mathbb{C}[x, y]$ such that $G = G_{\mathbb{C}}(f, g)$ has girth eight and G is not isomorphic to $G_{\mathbb{C}}(xy, xy^2)$?

1. This is done when g is a monomial (i.e. when $g = \alpha x^k y^m$ for $\alpha \in \mathbb{C}$).
2. G has a 4-cycle if and only if there exists a solution a, b, x, y to the system
$$\begin{cases} f(a,x) - f(b,x) + f(b,y) - f(a,y) = 0 \\ g(a,x) - g(b,x) + g(b,y) - g(a,y) = 0 \\ a \neq b, \;\; x \neq y. \end{cases}$$
3. G has a 6-cycle if and only if there exists a solution a, b, c, x, y, z to the system
$$\begin{cases} f(a,x) - f(b,x) + f(b,y) - f(c,y) + f(c,z) - f(a,z) = 0 \\ g(a,x) - g(b,x) + g(b,y) - g(c,y) + g(c,z) - g(a,z) = 0 \\ a \neq b, \;\; b \neq c, \;\; a \neq c; \;\; x \neq y, \;\; y \neq z, \;\; x \neq z. \end{cases}$$