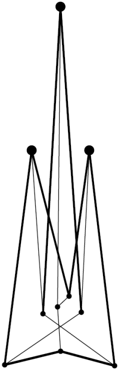
Switched symplectic graphs and their 2-rank

Aida Abiad

joint work with Willem H. Haemers

Tilburg University

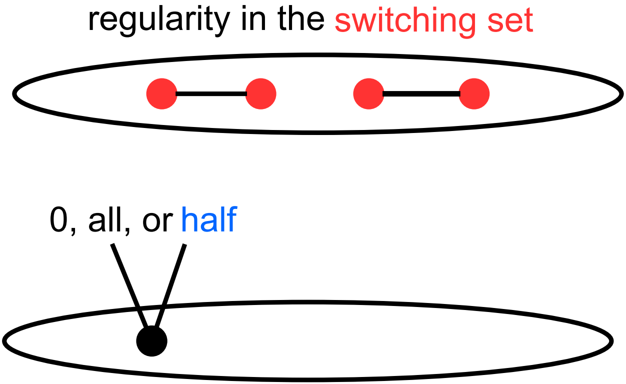
A church, a graph, and its adjacency matrix



$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

spectrum: $3, 1, 1, 1, 1, 1, -2, -2, -2, -2$

Godsil-McKay switching



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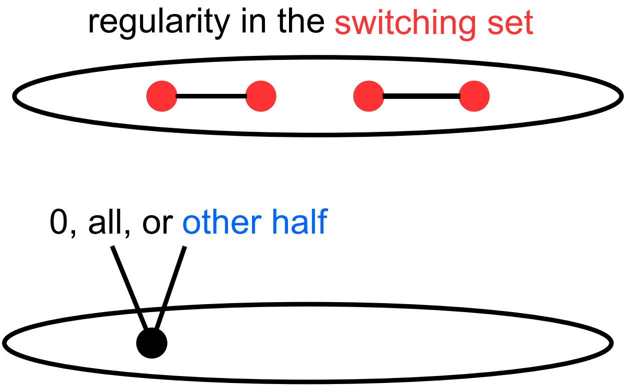

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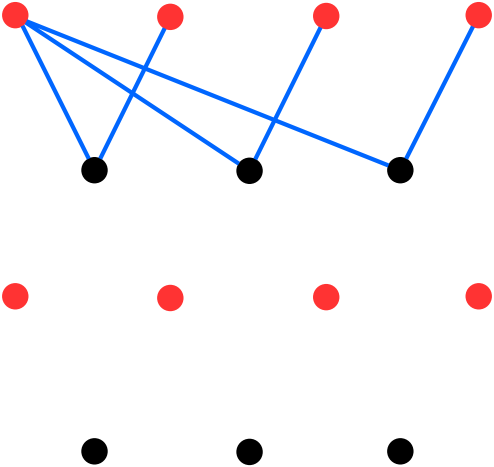

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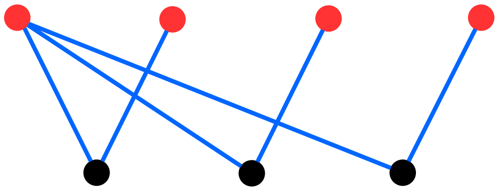

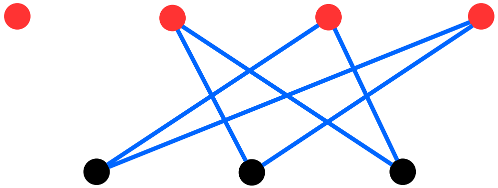

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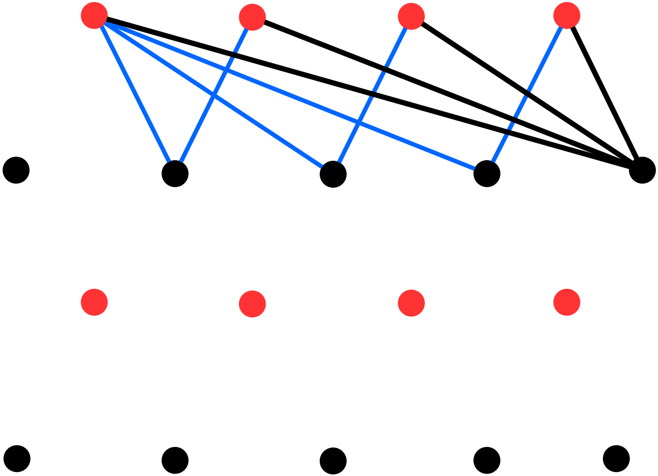

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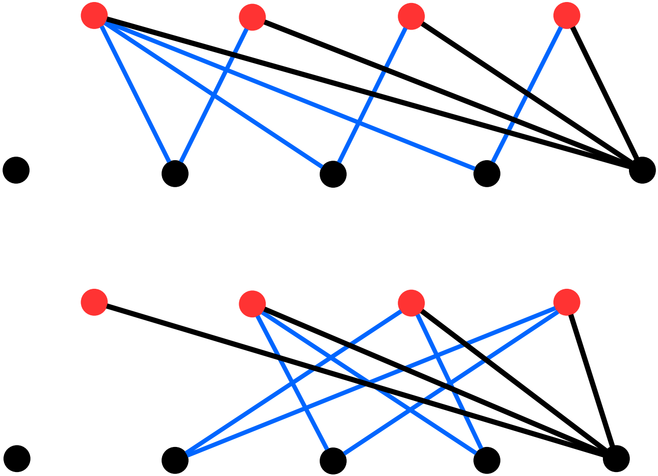

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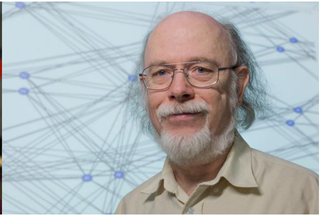

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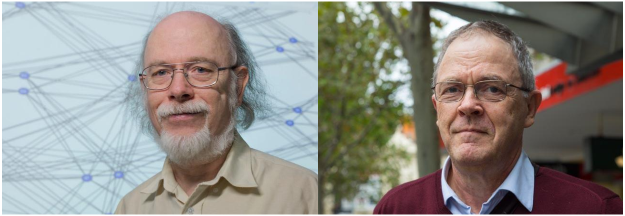

Switched symplectic graphs and their 2-ranks

Aida Abiad and Willem H. Haemers
Department of Econometrics and Operations Research,
Tilburg University, Tilburg, The Netherlands

The 2-rank of a graph is the rank of its adjacency matrix over the finite field $\mathbb{F}_2$.

Godsil-McKay switching and its 2-rank behaviour

$$A = \left[\begin{array}{c|c|c|c} B & N & J & O \\ \hline N^\top & \multicolumn{3}{c}{} \\ \cline{1-1} J & \multicolumn{3}{c}{C} \\ \cline{1-1} O & \multicolumn{3}{c}{} \end{array}\right] \qquad A' = \left[\begin{array}{c|c|c|c} B & \bar{N} & J & O \\ \hline \bar{N}^\top & \multicolumn{3}{c}{} \\ \cline{1-1} J & \multicolumn{3}{c}{C} \\ \cline{1-1} O & \multicolumn{3}{c}{} \end{array}\right]$$

$$S = \left[\begin{array}{c|c|c|c} O & J & O & O \\ \hline J & \multicolumn{3}{c}{} \\ \cline{1-1} O & \multicolumn{3}{c}{O} \\ \cline{1-1} O & \multicolumn{3}{c}{} \end{array}\right]$$

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$$S = \left[\begin{array}{c|ccc} O & J & O & O \\ \hline J & & & \\ O & & O & \\ O & & & \end{array}\right]$$

$$A' = A + S \pmod 2$$

Godsil-McKay switching and its 2-rank behaviour

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Lemma

The 2-rank of a symmetric integral matrix with zero diagonal is even.

Godsil-McKay switching and its 2-rank behaviour

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Proposition [Abiad and Haemers, 2015]

Suppose 2-rank$(A) = r$, then r is even and 2-rank$(A') = r-2$, r or $r+2$.

Strongly regular graph (SRG)

A regular graph or order n and degree k is *strongly regular* with parameters (n, k, λ, μ) if

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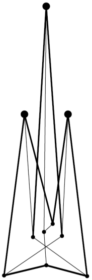
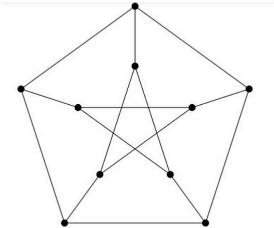
A regular graph or order n and degree k is *strongly regular* with parameters (n, k, λ, μ) if

- i Any two adjacent vertices have exactly λ common neighbours
- ii Any two nonadjacent vertices have exactly μ common neighbours

Example: Petersen graph is SRG$(10, 3, 0, 1)$





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G': graph obtained from G by Godsil-McKay switching
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G' is also SRG with the same parameters as G

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OUR TOOL: 2-rank

Symplectic graph over $\mathbb{F}_2$: $Sp(2\nu, 2)$

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$(v_1 w_2 + v_2 w_1) + (v_3 w_4 + v_4 w_3) + \cdots = 1$ iff

$v^\top K w = 1$ where

$$K = \begin{pmatrix} \begin{matrix} & 1 \\ 1 & \end{matrix} & & & \\ & \begin{matrix} & 1 \\ 1 & \end{matrix} & & \\ & & \ddots & \\ & & & \begin{matrix} & 1 \\ 1 & \end{matrix} \end{pmatrix}$$

The symplectic graph $Sp(2\nu, 2)$ is a SRG with parameters

$$\left(2^{2\nu} - 1,\ 2^{2\nu-1},\ 2^{2\nu-2},\ 2^{2\nu-2}\right).$$

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Theorem [Peeters, 1995]
$Sp(2\nu, 2)$ is characterized by its parameters and the minimality of its 2-rank, which equals 2ν.

Godsil-McKay switching set

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ z \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ z \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ z \end{bmatrix}, v_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ z \end{bmatrix}$$

where $z \in \mathbb{F}_2^{2\nu-6}$.

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Proposition [Abiad and Haemers, 2015]
The set $B = \{v_1, v_2, v_3, v_4\}$ is a Godsil-McKay switching set of $Sp(2\nu, 2)$ for $\nu \geq 3$.

Proof

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Any two vertices from B are nonadjacent

$$\Downarrow$$

subgraph induced by B is a coclique

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2. Consider $x \notin B$. Then

$$x^\top K v_1 + x^\top K v_2 + x^\top K v_3 + x^\top K v_4 = x^\top K(v_1 + v_2 + v_3 + v_4) = x^\top K \mathbf{0} = 0.$$

Theorem [Abiad and Haemers, 2015]
For $\nu \geq 3$, the graph G' obtained from $Sp(2\nu, 2)$ by switching with respect to the switching set B given above, is strongly regular with the same parameters as $Sp(2\nu, 2)$, but with 2-rank equal to $2\nu + 2$.

Sketch of the proof

- A adjacency matrix of $Sp(2\nu, 2)$.

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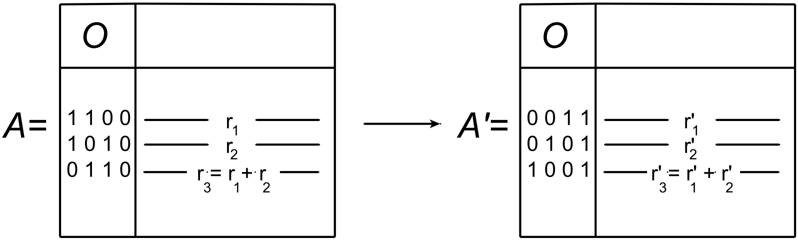
- The sum of any two rows of A is again a row of A.



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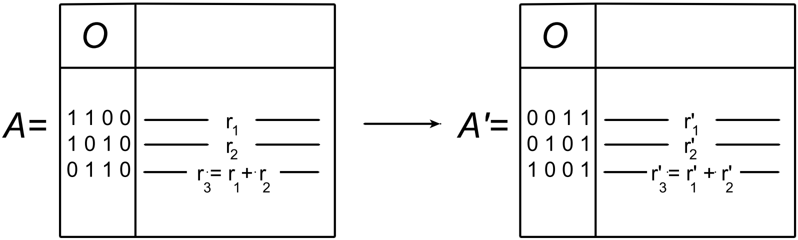

- Before switching: $r_1 + r_2 + r_3 = 00000\ldots0.$

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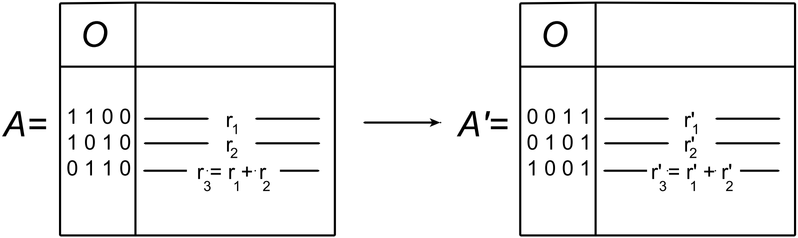

- Before switching: $r_1 + r_2 + r_3 = 00000\ldots0$.
- After switching: $r_1' + r_2' + r_3' = 11110\ldots0$.

Repeated switching in $Sp(6,2)$ gives $\geq$ 1000 nonisomorphic SRG $(63, 32, 16, 16)$ with 2-ranks:

$$6, 8, \ldots, 18.$$

Steve Butler is running an exhaustive search of repeated switching in $Sp(6,2)$ and has found ≥ 6000000 nonisomorphic SRG $(63,32,16,16)$ with 2-ranks:

$$6, 8, \ldots, 24.$$

Remark

Theoretical upper bound for the 2-rank of $Sp(6,2)$ is 26.

Hadamard matrix H of order n:

$$HH^\top = nI$$

Example:

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & - & - \\ 1 & - & 1 & - \\ 1 & - & - & 1 \end{bmatrix}$$

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Normalized Hadamard matrix: first row and first column is all ones.

Graphical Hadamard matrix: symmetric and constant diagonal.

For a Hadamard matrix H: $\quad A_H = \frac{1}{2}(J - H)$.

If H is normalized and $n > 4$,

$$H = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & & & \\ \vdots & & \ddots & & \\ 1 & & & 1 & \\ 1 & & & & 1 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & & & \\ \vdots & & \ddots & & \\ 0 & & & 0 & \\ 0 & & & & 0 \end{bmatrix}$$

and A corresponds to a $\mathrm{SRG}\left(n-1, \frac{n}{2}, \frac{n}{4}, \frac{n}{4}\right)$.

H_1, H_2 Hadamard matrices $\Longrightarrow$ $H_1 \otimes H_2$ Hadamard matrix.

Lemma [Abiad and Haemers, 2015]
Let H_1 and H_2 be Hadamard matrices, and let $\rho(H) = \text{2-rank}(A_H)$. Then,

$$\rho(H_1 \otimes H_2) \leq \rho(H_1) + \rho(H_2),$$

with equality if H_1 and H_2 are normalized.

Alternative description of $Sp(2\nu, 2)$ by using a recursive construction

Take

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & - & - \\ 1 & - & 1 & - \\ 1 & - & - & 1 \end{bmatrix}, \qquad \text{then} \qquad A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \Longleftarrow Sp(2,2)$$

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with 2-rank$(A) = 2$. We define

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$$H^{\otimes \nu} = H \otimes H \otimes \cdots \otimes H \quad (\nu \text{ times})$$

which is a normalized graphical Hadamard matrix of order 4^{ν} and 2-rank$(A_{H^{\otimes \nu}}) = 2\nu$.

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Theorem [Peeters, 1995]
$Sp(2\nu, 2)$ is characterized by its parameters and the minimality of its 2-rank, which equals 2ν.

The SRG associated with $H^{\otimes\nu}$ is $Sp(2\nu, 2)$

In $H^{\otimes\nu}$, we can replace any $H \otimes H \otimes H$ by any other regular graphical Hadamard matrix of order 64 coming from the SRG of order 63 found by computer (with 2-ranks 6,8,10,12,14,16,18).

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Theorem [Abiad and Haemers, 2015]
Using the above recursive construction we get SRG with the parameters of $Sp(2\nu, 2)$ and 2-ranks:

$$2\nu,\ 2\nu + 2, \ldots, 2\nu + 12\lfloor \nu/3 \rfloor.$$

Twisted symplectic polar graphs and Gordon-Mills-Welch difference sets

Akihiro Munemasa[1]
(joint work with Frédéric Vanhove)

[1]Graduate School of Information Sciences
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February 28, 2014
Colloquium on Galois Geometry
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A different construction of graphs with the same parameters as $Sp(2\nu, 2)$ was given by Munemasa and Vanhove in 2014. The construction admits a cyclic difference set, and using a result by Qing et al., it follows that the 2-rank of their graphs is at least 4ν

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$\Downarrow$

Nonisomorphic to our switched symplectic graphs, which have 2-rank $2\nu + 2$

Summary

- We find Godsil-McKay switching sets in $Sp(2\nu, 2)$ and we prove that the 2-rank increases after switching
- Repeated switching in $Sp(6, 2)$
- Recursive construction method for $Sp(2\nu, 2)$ from Hadamard matrices

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- Repeated switching in $Sp(6, 2)$
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New SRG with the same parameters as $Sp(2\nu, 2)$, but different 2-ranks

- Are there superexponentially many switched symplectic graphs obtained by repeated Godsil-McKay switching on $Sp(2\nu, 2)$?

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- Are the twisted symplectic polar graphs (with 2-rank at least 4ν) related to the switched symplectic graphs (with 2-rank $2\nu + 2$)?

Thank you for your attention.

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And thanks to the organizer!