MATH 676

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Finite element methods in scientific computing

Wolfgang Bangerth, Texas A&M University

Lecture 38:

What preconditioner to use

Part 5: Complex ("physics-based"/"block") preconditioners for complex problems

Part 5:
Constructing complex preconditioners
for non-scalar, non-trivial problems

(late 1990s – today)

“Physics-based preconditioners”

1990s: How to solve time-dependent, coupled problems?

Example: Thermoelasticity

$$
\begin{aligned}
\frac{\partial T}{\partial t} - \kappa \Delta T &= q + \epsilon(\dot{\vec{u}}) : C \epsilon(\dot{\vec{u}}) \\
-\lambda \nabla (\nabla \cdot \vec{u}) - \mu \nabla \cdot (\nabla \vec{u} + \nabla \vec{u}^T) &= \beta \nabla T
\end{aligned}
$$

In time step *n*, this leads to a problem of the form

$$
\begin{pmatrix} M + \Delta t\ A & -E \\ -B & C \end{pmatrix} \begin{pmatrix} T^n \\ U^n \end{pmatrix} = \begin{pmatrix} F^n \\ 0 \end{pmatrix}
$$

Approach: In many problems, *B*, *E* are small. At least *E* is.

"Physics-based preconditioners"

1990s: How to solve time-dependent, coupled problems?

Example: If the problem is weakly coupled, then

$$\begin{pmatrix} M+\Delta t\ A & -E \\ -B & C \end{pmatrix} \approx \begin{pmatrix} M+\Delta t\ A & 0 \\ 0 & C \end{pmatrix}$$

and a good preconditioner would be

$$P^{-1} = \begin{pmatrix} M+\Delta t\ A & 0 \\ 0 & C \end{pmatrix}^{-1} = \begin{pmatrix} (M+\Delta t\ A)^{-1} & 0 \\ 0 & C^{-1} \end{pmatrix}$$

“Physics-based preconditioners”

Question: How to apply the preconditioner

$$P^{-1} = \begin{pmatrix} M+\Delta t\ A & 0 \\ 0 & C \end{pmatrix}^{-1} = \begin{pmatrix} (M+\Delta t\ A)^{-1} & 0 \\ 0 & C^{-1} \end{pmatrix}$$

Answer: Multiplying with it is equivalent to this:

$$\begin{pmatrix} x^T_{\text{pre}} \\ x^u_{\text{pre}} \end{pmatrix} = P^{-1} \begin{pmatrix} x^T \\ x^u \end{pmatrix} \quad \Leftrightarrow \quad \begin{pmatrix} (M+\Delta t\ A)\, x^T_{\text{pre}} \\ C\, x^u_{\text{pre}} \end{pmatrix} = \begin{pmatrix} x^T \\ x^u \end{pmatrix}$$

- Preconditioning means solving one timestep for temperature and elasticity independently
- This is why we call it “physics-based”
- We typically have good solvers for each “physics”

“Physics-based preconditioners”

1990s: How to solve time-dependent, coupled problems?

Example: If the problem is weakly coupled, then

$$\begin{pmatrix} M+\Delta t\ A & -E \\ -B & C \end{pmatrix} \approx \begin{pmatrix} M+\Delta t\ A & 0 \\ 0 & C \end{pmatrix}$$

and a *better* preconditioner would be either

$$P^{-1} = \begin{pmatrix} M+\Delta t\ A & 0 \\ -B & C \end{pmatrix}^{-1} = \begin{pmatrix} (M+\Delta t\ A)^{-1} & 0 \\ -(M+\Delta t\ A)^{-1} B & C^{-1} \end{pmatrix}$$

or

$$P^{-1} = \begin{pmatrix} M+\Delta t\ A & -E \\ 0 & C \end{pmatrix}^{-1} = \begin{pmatrix} (M+\Delta t\ A)^{-1} & -C^{-1} E \\ 0 & C^{-1} \end{pmatrix}$$

Note: Choose the one that includes the stronger coupling.

"Physics-based preconditioners"

Question: How to apply the preconditioner

$$P^{-1} = \begin{pmatrix} M+\Delta t\ A & -E \\ 0 & C \end{pmatrix}^{-1} = \begin{pmatrix} (M+\Delta t\ A)^{-1} & -C^{-1}E \\ 0 & C^{-1} \end{pmatrix}$$

Answer: Multiplying with it is equivalent to this:

$$\begin{pmatrix} x^T_{\text{pre}} \\ x^u_{\text{pre}} \end{pmatrix} = P^{-1} \begin{pmatrix} x^T \\ x^u \end{pmatrix} \quad \Leftrightarrow \quad \begin{pmatrix} M+\Delta t\ A & -E \\ 0 & C \end{pmatrix} \begin{pmatrix} x^T_{\text{pre}} \\ x^u_{\text{pre}} \end{pmatrix} = \begin{pmatrix} x^T \\ x^u \end{pmatrix}$$

That is:

$$\begin{aligned} \text{step 1:} \qquad & C\, x^u_{\text{pre}} &&= x^u \\ \text{step 2:} \qquad & (M+\Delta t\ A)\, x^T_{\text{pre}} &&= x^T + E\, x^u_{\text{pre}} \end{aligned}$$

Note: This is exactly the same effort as before!

“Physics-based preconditioners”

Basic insights:

- These preconditioners are often *really good*!
- In particular if coupling is primarily one-way
- “Block preconditioners” are often much better than “point preconditioners” (e.g. Vanka)
- Can be generalized to problems with more than two “physics”

“Physics-based” vs. “block” preconditioners

Question:

Can we use these insights for
single-physics, coupled equations?

Example: Stokes

$$\begin{aligned} -\mu \Delta u + \nabla p &= f \\ \nabla \cdot u &= 0 \end{aligned} \qquad \begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} U \\ P \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix}$$

“Physics-based” vs. “block” preconditioners

Answer: Yes!

Also:

The key to this is understanding
the *Schur* complement.

Schur complement

Example: Stokes

$$\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} U \\ P \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix}$$

Here, *A* is an invertible matrix. Consequently:

$$\begin{array}{rcl} AU+BP & = & F \\ B^T U & = & 0 \end{array} \quad \Rightarrow \quad \begin{array}{rcl} U+A^{-1}BP & = & A^{-1}F \\ B^T U & = & 0 \end{array} \quad \Rightarrow \quad \begin{array}{rcl} U+A^{-1}BP & = & A^{-1}F \\ B^T A^{-1}BP & = & B^T A^{-1}F \end{array}$$

Note:

- We call $S=B^TA^{-1}B$ the *Schur complement* of the matrix
- We obtained *S* by block Gauss elimination

Schur complement

Application: We could solve the Stokes system

$$\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} U \\ P \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix}$$

by solving in the following two (decoupled) steps:

$$\begin{aligned} SP &= B^T A^{-1} F \\ AU &= F - BP \end{aligned}$$

Problem:

- We do not have $S = B^T A^{-1} B$ element-by-element
- S is in fact a dense matrix
- However, we could take it as an operator (see step-20/22)

Schur complement

Application: We could solve the Stokes system

$$\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} U \\ P \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix}$$

by solving in the following two (decoupled) steps:

$$\begin{aligned} SP &= B^T A^{-1} F \\ AU &= F - BP \end{aligned}$$

Insight: This two-step procedure corresponds to

$$\begin{pmatrix} A & B \\ 0 & S \end{pmatrix} \begin{pmatrix} U \\ P \end{pmatrix} = \begin{pmatrix} F \\ B^T A^{-1} F \end{pmatrix}$$

$$\Rightarrow \qquad \begin{pmatrix} U \\ P \end{pmatrix} = \begin{pmatrix} A & B \\ 0 & S \end{pmatrix}^{-1} \begin{pmatrix} F \\ B^T A^{-1} F \end{pmatrix} = \begin{pmatrix} A^{-1} & -A^{-1} B S^{-1} \\ 0 & S^{-1} \end{pmatrix} \begin{pmatrix} F \\ B^T A^{-1} F \end{pmatrix}$$

Schur complement

Idea: This suggests that

$$P^{-1} = \begin{pmatrix} A^{-1} & -A^{-1} B S^{-1} \\ 0 & S^{-1} \end{pmatrix}$$

might be a good preconditioner for

$$\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix}$$

Indeed:

$$\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} A^{-1} & -A^{-1} B S^{-1} \\ 0 & S^{-1} \end{pmatrix} = \begin{pmatrix} I & 0 \\ B^T A^{-1} & I \end{pmatrix}$$

Schur complement

Idea: This suggests that

$$P^{-1} = \begin{pmatrix} A^{-1} & -A^{-1}BS^{-1} \\ 0 & S^{-1} \end{pmatrix}$$

might be a good preconditioner for

$$\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix}$$

Notes:

- One can show that with this preconditioner, GMRES converges in two steps (Silvester & Wathen, 1994)
- "Theoretical" block preconditioner: We do not have S!

Approximate Schur complement

Idea 2: If

$$P^{-1} = \begin{pmatrix} A^{-1} & -A^{-1}BS^{-1} \\ 0 & S^{-1} \end{pmatrix}$$

is not feasible, then maybe something like

$$P^{-1} = \begin{pmatrix} \widetilde{A^{-1}} & -\widetilde{A^{-1}}B\widetilde{S^{-1}} \\ 0 & \widetilde{S^{-1}} \end{pmatrix}$$

could work!

Here:

- Tildes indicate approximate inverse operators
- For example for *A*: do one SSOR step with *A*, or one multigrid step

Approximate Schur complement

Question: What to do with S?

Some answers (Silvester & Wathen, 1994):

Recall that

- $S = B^T A^{-1} B$
- $B \simeq \mathrm{grad}$
- $B^T \simeq -\mathrm{div}$
- $A \simeq -\Delta$

Thus: One may think that

$$S = B^T A^{-1} B \quad \simeq \quad -\mathrm{div}\,(-\Delta)^{-1}\,\mathrm{grad} = -\mathrm{div}\,\mathrm{grad}\,(-\Delta)^{-1} = \mathrm{Id}$$

S might be close to the mass matrix, so maybe $\widetilde{S}^{-1} = M_p^{-1}$?

Approximate Schur complement

Some more answers:

- The replacement $\widetilde{S}^{-1} = M_p^{-1}$ indeed leads to a pretty good preconditioner
- See “results” section of step-22 for implementation and results

However: The reasoning

$$S = B^T A^{-1} B \qquad \simeq \qquad -\mathrm{div}\left(-\Delta\right)^{-1}\mathrm{grad} = -\mathrm{div}\,\mathrm{grad}\left(-\Delta\right)^{-1} = \mathrm{Id}$$

is flawed and wrong because the operators do not commute!

(But: it works anyway, and there are good reasons for that.)

Block preconditioners

Summary: Since the late 1990s, we have learned:

- Good preconditioners can be constructed by playing with the blocks of a couple system matrix
- “Small” off-diagonal blocks for weak influences may be dropped
- Invertible, known diagonal blocks can be exactly solved
- Invertible Schur complements on the diagonal can often be approximated
  (see step-20, step-22, step-31, ...)

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