MATH 676

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Finite element methods in scientific computing

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Lecture 33:

Which element to use
Part 1: “Simple” problems

Elements

What we've seen so far:

- Steps 1 – 6 (Laplace):
  - scalar equation
  - Q1 or Q2 elements
  - easy to change
- Step 20 (mixed Laplace):
  - vector-valued equation
  - Raviart-Thomas element for the velocity
  - piecewise constants for pressure (or higher order DG)
  - pairing needs to satisfy certain conditions
- Step 22 (Stokes):
  - vector-valued equation
  - Q2 element for the velocity
  - Q2 for pressure
  - pairing needs to satisfy certain conditions

Elements

There is a zoo of elements for different purposes:

- Continuous Lagrange
- Discontinuous Lagrange
- Raviart-Thomas
- Nedelec
- Rannacher-Turek
- Brezzi-Douglas-Marini (BDM)
- Brezzi-Douglas-Duran-Marini (BDDM)
- Hermite (Argyris)
- Crouzeix-Raviart
- Arnold-Falk-Winther
- Arnold-Boffi-Falk (ABF)

  …
- Hybridized elements
- Penalized discontinuous elements

Elements

There is a zoo of elements for different purposes:

• Continuous Lagrange	FE_Q
• Discontinuous Lagrange	FE_DGQ, FE_DGP
• Raviart-Thomas	FE_RaviartThomas
• Nedelec	FE_Nedelec
• Rannacher-Turek	
• Brezzi-Douglas-Marini (BDM)	FE_BDM
• Brezzi-Douglas-Duran-Marini (BDDM)	
• Hermite (Argyris)	
• Crouzeix-Raviart	
• Arnold-Falk-Winther	
• Arnold-Boffi-Falk (ABF)	FE_ABF
...	
• Hybridized elements	FE_FaceQ/TraceQ
• Penalized discontinuous elements	

Scalar problems

For scalar problems like the Laplace equation:

- Q_p elements are generally the right choice
- Higher *p* yield higher convergence order for elliptic problems:

$$\|u-u_h\|_{H^1} \leq Ch^p|u|_{H^{p+1}} \qquad \|u-u_h\|_{L_2} \leq Ch^{p+1}|u|_{H^{p+1}}$$

- Number of degrees of freedom grows as:

$$N \simeq \frac{|\Omega|}{(h/p)^d} = p^d\frac{|\Omega|}{h^d} \quad \rightarrow \quad h \simeq p\left(\frac{|\Omega|}{N}\right)^{1/d}$$

- Error as function of *N*:

$$\|u-u_h\|_{H^1} \simeq p^p N^{-p/d} \qquad \|u-u_h\|_{L_2} \simeq p^{p+1} N^{-(p+1)/d}$$

Consequence: This suggests high order elements!

Scalar problems

For scalar problems like the Laplace equation:

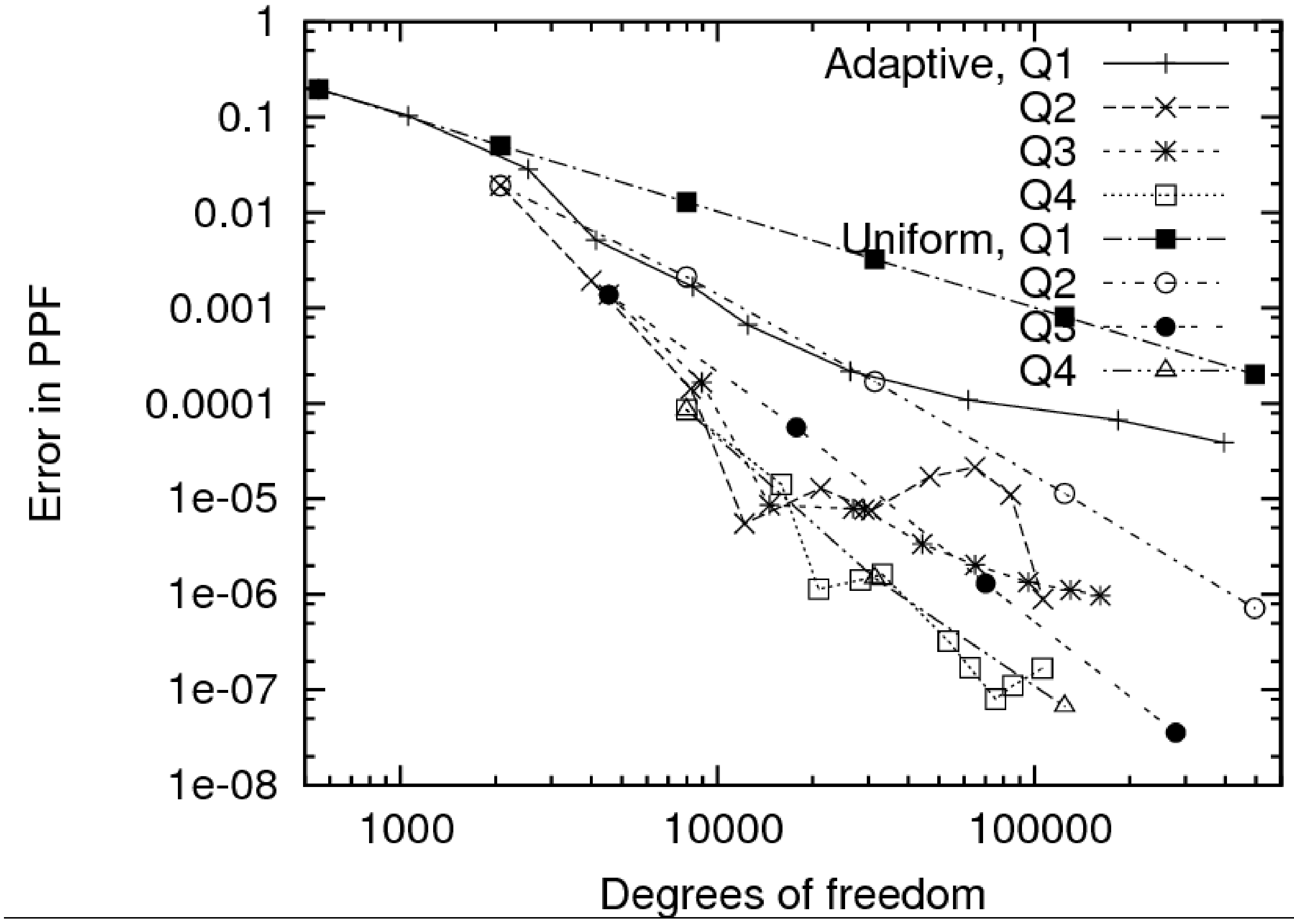
- Q_p elements are generally the right choice
- Better convergence only if *u* smooth: $\|u-u_h\|_{H^1} \leq Ch^p|u|_{H^{p+1}}$
- Higher *p* also requires more work:
  – more computations to assemble matrix: $O(p^d)$
  – more entries per row in the matrix: $O(p^d)$
  – good preconditioners are difficult to construct for high *p*

Consequence: This suggests low order elements!

Together: It is a trade-off!

Practical experience

Prototypical 2d example from Wang, Bangerth, Ragusa (2007, Progress in Nuclear Energy):



Practical experience

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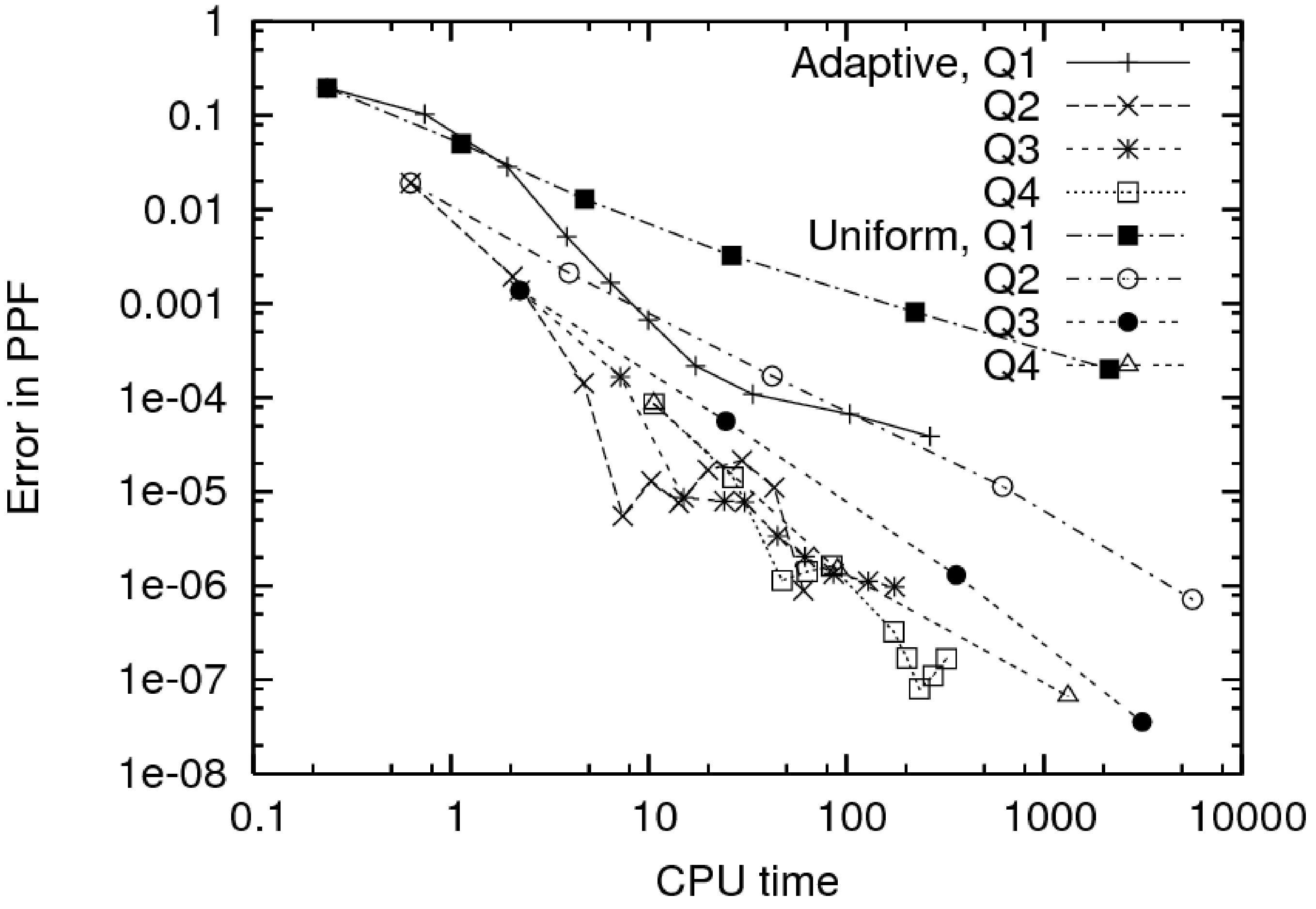


Practical experience

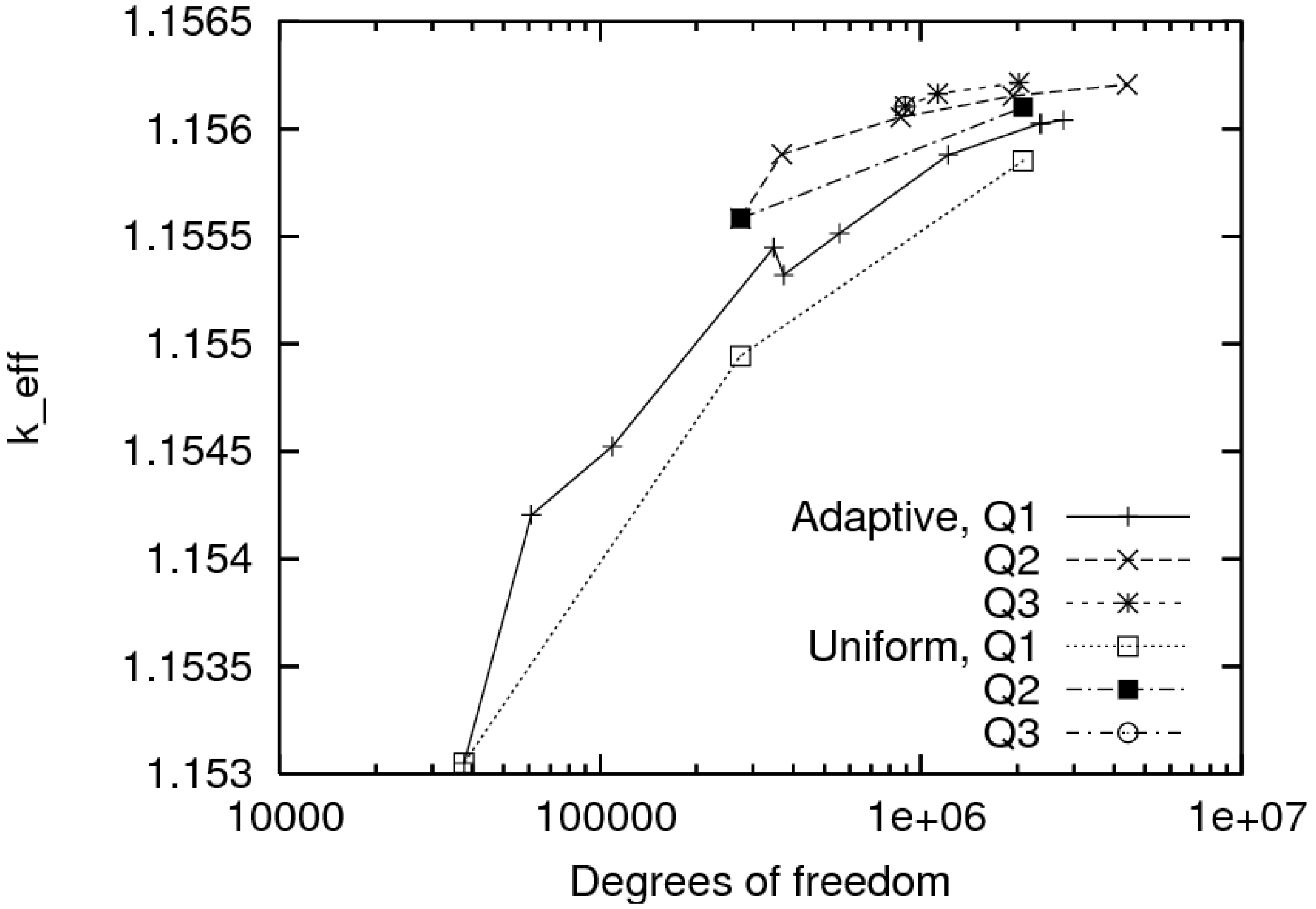
Prototypical 2d example from Wang, Bangerth, Ragusa (2007, Progress in Nuclear Energy):

Conclusions:

- Higher p gives better error-per-dof
- Not so clear any more for error-per-CPU-second
- Sweat spot maybe around $p=3$ or $p=4$ in 2d

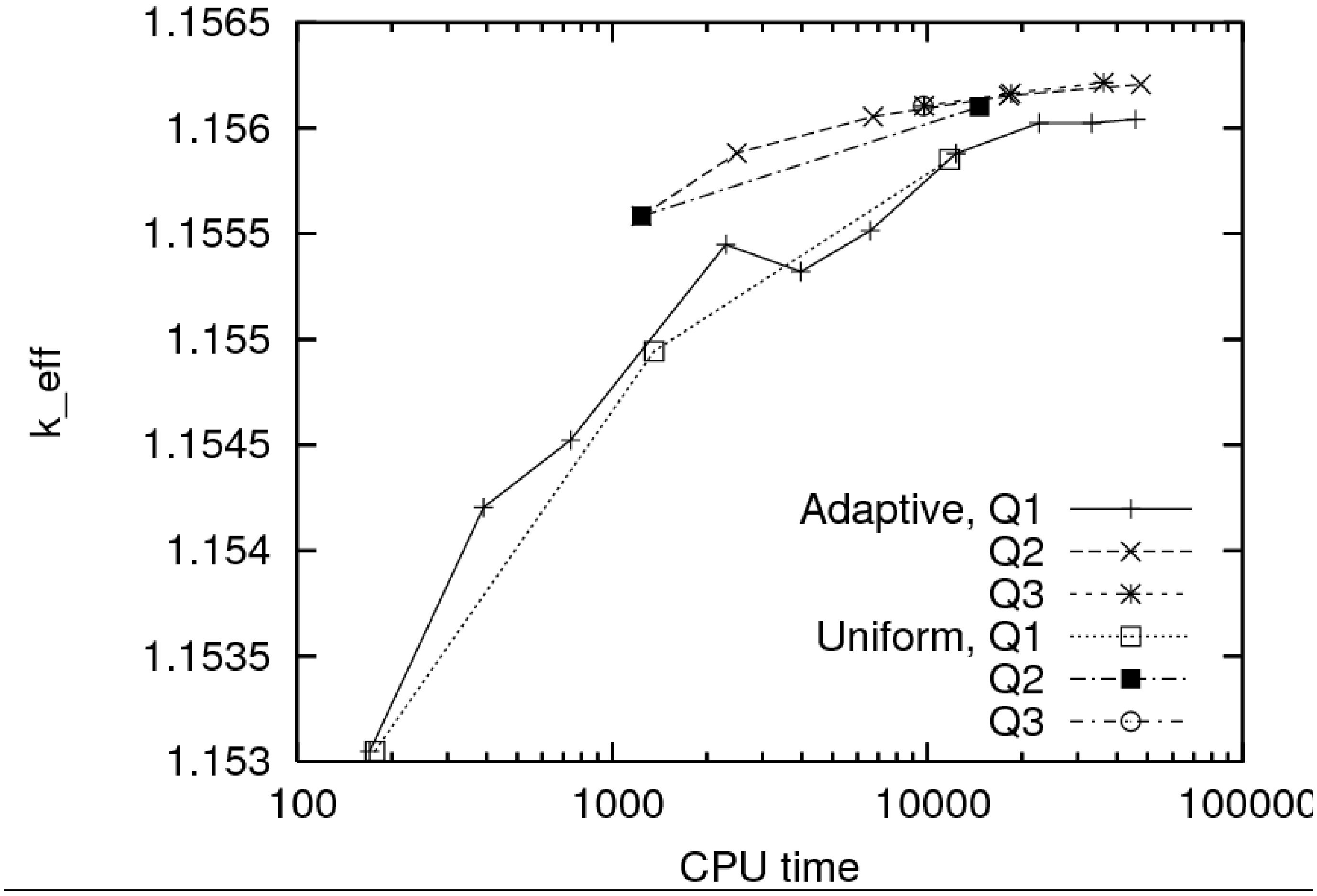
Practical experience

Prototypical 3d example from Wang, Bangerth, Ragusa (2007, Progress in Nuclear Energy):



Practical experience

Prototypical 3d example from Wang, Bangerth, Ragusa (2007, Progress in Nuclear Energy):



Practical experience

Prototypical 3d example from Wang, Bangerth, Ragusa (2007, Progress in Nuclear Energy):

Conclusions:

- Higher p gives better error-per-dof
- Not so clear any more for error-per-CPU-second

- Sweat spot maybe around $p=2$ or $p=3$ in 3d

Practical experience

Conclusions for scalar problems:

- There is a trade-off between faster convergence and more work

- A good compromise is:
  – Q3 or Q4 in 2d
  – Q2 or Q3 in 3d

Electromagnetics

A simple vector-valued equation:

- Consider the Maxwell equations:

$$\text{curl } B = j + \frac{\partial E}{\partial t}$$
$$\text{div } B = 0$$
$$\text{curl } E = -\frac{\partial B}{\partial t}$$
$$\text{div } E = q$$

- If *j=0*, *q=0*, we can decouple these equations:

$$\frac{\partial^2 B}{\partial t^2} + \text{curl curl } B = 0$$
$$\text{div } B = 0$$
$$\frac{\partial^2 E}{\partial t^2} + \text{curl curl } E = 0$$
$$\text{div } E = 0$$

Electromagnetics

The source-free Maxwell equations:

In the equations

$$\frac{\partial^2 B}{\partial t^2} + \text{curl curl } B = 0$$
$$\text{div } B = 0$$
$$\frac{\partial^2 E}{\partial t^2} + \text{curl curl } E = 0$$
$$\text{div } E = 0$$

each variable $u \in \{E, B\}$ satisfies an equation of the form

$$\frac{\partial^2 u}{\partial t^2} + \text{curl curl } u = 0$$
$$\text{div } u = 0$$

Electromagnetics

The source-free Maxwell equations:

Consider the time-independent case for simplicity:

$$\begin{aligned} \text{curl}\ \text{curl}\ u &= 0 \\ \text{div}\ u &= 0 \end{aligned}$$

The “simplest” variational formulation would use the weak form

$$(\text{curl}\ v, \text{curl}\ u) + (\text{div}\ v, \text{div}\ u) = 0 \qquad \forall v$$

This requires solutions $u \in \underbrace{H_{\text{curl}} \cap H_{\text{div}}}_{=:V} \supset H^1$

Electromagnetics

The source-free Maxwell equations:

One might think that we can approximate solutions of

$$(\text{curl } v, \text{curl } u) + (\text{div } v, \text{div } u) = 0 \qquad \forall v$$

using the usual Lagrange (Q_p) elements.

However, not so:

- The Lagrange (Q_p) element space is $V_h \subset H^1 \subset V$
- H_1 is not dense in V with respect to the norm $\|\cdot\|_V = \|\cdot\|_{H_{\text{curl}} \cap H_{\text{div}}}$
- We may not converge to the correct solution

[Lack of denseness: Costabel 1991]

Electromagnetics

The source-free Maxwell equations:

One might think that we can approximate solutions of

$$(\mathrm{curl}\ v, \mathrm{curl}\ u) + (\mathrm{div}\ v, \mathrm{div}\ u) = 0 \qquad \forall v$$

using the usual Lagrange (Q_p) elements.

Alternative:

- Use Nedelec finite elements where $V_h \notin H^1, \ V_h \subset V$
- $\lim_{h\to 0} V_h$ is dense in V with respect to the norm $\|\cdot\|_V$
- We converge to the correct solution

Electromagnetics

Source-free Maxwell equations summary:

- Use Nedelec finite elements (FE_Nedelec)

- In practice, people typically use lowest order elements
- This may be a mistake:
  - Probably better performance for $k=2...4$
  - Higher order Nedelec elements difficult to implement

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