MATH 676

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Finite element methods in scientific computing

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Lecture 31.65:

Nonlinear problems

Part 4: Fixed point/Picard iteration for the minimal surface equation

The minimal surface equation

Consider the minimal surface equation:

$$\begin{aligned} -\nabla \cdot \left(\frac{A}{\sqrt{1+|\nabla u|^2}} \nabla u \right) &= f \qquad \text{in } \Omega \\ u &= g \qquad \text{on } \partial\Omega \end{aligned}$$

where we choose

$$\Omega = B_1(0) \subset \mathbb{R}^2, \qquad f = 0, \qquad g = \sin(2\pi(x+y))$$

Goal: Solve this numerically with a Picard iteration method.

Picard iteration

General approach: To solve

$$L(u) = f$$

by Picard iteration, we need to rewrite *L(u)* as

$$L(u) = G(u)u$$

and then repeatedly solve

$$G(u_k)u_{k+1} = f$$

Note 1: Here, *G(u)* is in general a differential operator.
Note 2: This is a linear problem in u_{k+1}.

Picard iteration

Here: To solve

$$-\nabla\cdot\left(\frac{A}{\sqrt{1+|\nabla u|^2}}\nabla u\right) = f \quad \text{in } \Omega$$
$$u = g \quad \text{on } \partial\Omega$$

by Picard iteration, we need to repeatedly solve

$$-\nabla\cdot\left(\frac{A}{\sqrt{1+|\nabla u_k|^2}}\nabla u_{k+1}\right) = f \quad \text{in } \Omega$$
$$u_{k+1} = g \quad \text{on } \partial\Omega$$

Note: This is a linear problem in u_{k+1}.

Picard iteration

Question: With

$$L(u) = -\nabla\cdot\left(\frac{A}{\sqrt{1+|\nabla u|^2}}\nabla u\right)$$

why choose

$$G(u)w = -\nabla\cdot\left(\frac{A}{\sqrt{1+|\nabla u|^2}}\nabla w\right)$$

? We could also choose

$$G(u)w = \frac{L(u)}{u}w = \left[-\nabla\cdot\left(\frac{A}{\sqrt{1+|\nabla u|^2}}\nabla u\right)\right]\frac{w}{u}$$

Answer: Yes, but this is no differential equation in *w*. It may or may not converge.

Picard iteration

Question: With

$$L(u) = -\nabla\cdot\left(\frac{A}{\sqrt{1+|\nabla u|^2}}\nabla u\right)$$

why choose

$$G(u)w = -\nabla\cdot\left(\frac{A}{\sqrt{1+|\nabla u|^2}}\nabla w\right)$$

? We could also choose

$$G(u)w = -\nabla\cdot\left(\frac{A}{\sqrt{1+\nabla u\cdot\nabla w}}\nabla u\right)$$

Answer: Yes, but this is not linear in *w*. We don't know how to solve $G(u)w = f$.

Picard iteration

Question: How to choose $G(u)$?

Answers:

- We often have different choices for $G(u)$
- We choose so that
  – $G(u)$ is linear
  – The equation $G(u)\, w = f$ is a well-posed PDE
  – The operator $G(u)$ has a “convenient” structure
- There is often an element of “experience” involved

Picard iteration

Remember: To solve

$$-\nabla \cdot \left(\frac{A}{\sqrt{1+|\nabla u|^2}} \nabla u \right) = f \quad \text{in } \Omega$$
$$u = g \quad \text{on } \partial\Omega$$

by Picard iteration, we need to repeatedly solve

$$-\nabla \cdot \left(\frac{A}{\sqrt{1+|\nabla u_k|^2}} \nabla u_{k+1} \right) = f \quad \text{in } \Omega$$
$$u_{k+1} = g \quad \text{on } \partial\Omega$$

Notes: (i) This is linear u_{k+1}. (ii) It is well-posed if the coefficient is bounded from below. (iii) It is “like” the Poisson equation.

Picard iteration

Picard iteration: Repeatedly solve

$$-\nabla \cdot \left(\frac{A}{\sqrt{1+|\nabla u_k|^2}} \nabla u_{k+1} \right) = f \quad \text{in } \Omega$$

$$u_{k+1} = g \quad \text{on } \partial\Omega$$

Question: What should our initial guess u_0 be?

Answers:

- As close as possible to the exact solution!
- Here, $u_0=0$ will suffice (even though this doesn't satisfy boundary conditions).

Picard iteration

Picard iteration: Repeatedly solve

$$-\nabla\cdot\left(\frac{A}{\sqrt{1+|\nabla u_k|^2}}\nabla u_{k+1}\right) = f \quad \text{in } \Omega$$
$$u_{k+1} = g \quad \text{on } \partial\Omega$$

Question: Wouldn't the solution of

$$-\nabla\cdot\left(A\nabla u_0\right) = f \quad \text{in } \Omega$$
$$u_0 = g \quad \text{on } \partial\Omega$$

be better?

Answer: Yes! But if $u_0=0$ then u_1 satisfies

$$-\nabla\cdot\left(\frac{A}{\sqrt{1+0^2}}\nabla u_1\right) = f \quad \text{in } \Omega$$
$$u_1 = g \quad \text{on } \partial\Omega$$

Picard iteration

Remember: To solve

$$-\nabla\cdot\left(\frac{A}{\sqrt{1+|\nabla u|^2}}\nabla u\right) = f \quad \text{in } \Omega$$
$$u = g \quad \text{on } \partial\Omega$$

by Picard iteration, we need to repeatedly solve

$$-\nabla\cdot\left(\frac{A}{\sqrt{1+|\nabla u_k|^2}}\nabla u_{k+1}\right) = f \quad \text{in } \Omega$$
$$u_{k+1} = g \quad \text{on } \partial\Omega$$

Let's see if we can morph step-6 to do this for us.

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