Finite element methods in scientific computing

Wolfgang Bangerth, Colorado State University

Lecture 3.95:

The ideas behind the finite element method

Part 6: Error estimates for the Laplace equation

Solutions

Recall (ignoring boundary values):

The "weak solution" $u \in H^1(\Omega)$ of the Laplace equation satisfies the "weak formulation":

$$(\nabla \varphi, \nabla u) = (\varphi, f) \qquad \forall \varphi \in H^1(\Omega)$$

The finite element solution $u_h \in V_h \subset H^1(\Omega)$ satisfies the "discrete weak formulation":

$$(\nabla \varphi_h, \nabla u_h) = (\varphi_h, f) \qquad \forall \varphi_h \in V_h \subset H^1(\Omega)$$

Here: V_h is the space of piecewise polynomial functions of degree p, defined on the mesh.

Previous questions

At the end of Lecture 3.92, we had 3 questions:

Question 3: Is the approximation u_h so defined "close" to the exact solution u?

Question 4: Does u_h "converge" towards u in some useful sense?

Question 5: What is the computational effort to reach a certain accuracy? Optimality?

Measuring errors

How do we *measure* whether *u* and u_h are close?

- We use a "norm" of $u-u_h$
- There are many possible norms:
  - Choice depends on the application
  - For some, estimating the error is easy, for others not.

Examples:

- Maximal (L^∞) error: $\|u-u_h\|_{L^\infty} = max_{x\in\Omega} |u(x)-u_h(x)|$
- Mean square (L^2) error: $\|u-u_h\|_{L^2} = \left(\int_\Omega |u(x)-u_h(x)|^2 \, dx\right)^{1/2}$
- Gradient (H^1, energy) error: $\|\nabla(u-u_h)\|_{L^2} = \left(\int_\Omega |\nabla(u(x)-u_h(x))|^2 \, dx\right)^{1/2}$

Form of error estimates

***A priori* error estimates (available *before* computing):**

Representations of the (relative) error of the form

$$\frac{\|u-u_h\|_?}{\|u\|_{?'}} \leq C(p)\; h^{\alpha(p)}$$

- Typically written as

$$\|u-u_h\|_? \leq C(p)\; h^{\alpha(p)}\; \|u\|_{?'}$$

- Guarantees convergence if $\alpha(p)>0$
- But absolute level of error unknown because *u* on the right hand side is unknown.

Form of error estimates

***A posteriori* estimates (available *after* computing):**

Representations of the (relative) error of the form

$$\|u-u_h\|_? \leq C(p)\, h^{\beta(p)}\, Q(u_h)$$

- Does not guarantee convergence even if $\beta(p)>0$
- Can only be evaluated after computing u_h
- Allows estimating actual size of the error
- Substantially more complicated to derive!
  (See Lecture 17.75)

Ingredient #1: Galerkin orthogonality

Starting point for most error estimation approaches:

- Discrete solution satisfies

$$(\nabla\varphi_h, \nabla u_h) = (\varphi_h, f) \qquad \forall \varphi_h \in V_h \subset H^1(\Omega)$$

- Exact solution satisfies:

$$(\nabla\varphi, \nabla u) = (\varphi, f) \qquad \forall \varphi \in H^1(\Omega)$$

- But because $V_h \subset H^1(\Omega)$, the exact solution also satisfies

$$(\nabla\varphi_h, \nabla u) = (\varphi_h, f) \qquad \forall \varphi_h \in V_h \subset H^1(\Omega)$$

Subtract the first from the third equation:

$$(\nabla\varphi_h, \nabla(u - u_h)) = 0 \qquad \forall \varphi_h \in V_h \subset H^1(\Omega)$$

Ingredient #1: Galerkin orthogonality

For finite element discretizations, we have:

$$\left(\nabla\varphi_h, \nabla(u-u_h)\right) = 0 \qquad \forall \varphi_h \in V_h \subset H^1(\Omega)$$

This is called **Galerkin orthogonality**:

The error $e=u-u_h$ is *perpendicular* to all elements $\varphi_h \in V_h$ with regard to the scalar product $(\nabla\circ, \nabla\circ)$!

Ingredient #2: Interpolation estimates

Recall (a variation of the theorem in) lecture 3.91:

$$\|\nabla(f - I_{h,p} f)\|_{L^2} \leq \frac{C(p,\Omega)}{p!} h^p \|\nabla^{p+1} f\|_{L^2}$$

Here:

- *f* can be any function that has sufficiently many derivatives
- $I_{h,p} f$ is the function that *interpolates f*:
  – on a mesh with maximal cell diameter *h*
  – has polynomial degree *p* on each cell

Estimating the gradient error

Start as follows:

$$\|\nabla(u-u_h)\|^2_{L^2} = \left(\nabla(u-u_h), \nabla(u-u_h)\right)$$

$$= \left(\nabla(u-u_h), \nabla(u-u_h)\right) + \underbrace{\left(\nabla\varphi_h, \nabla(u-u_h)\right)}_{=0\text{ (Galerkin orthogonality)}} \qquad \forall \varphi_h \in V_h$$

We can pick a "convenient" test function: $\varphi_h = u_h - I_{h,p}u \in V_h$

$$= \left(\nabla(u-I_{h,p}u), \nabla(u-u_h)\right)$$

Then apply the Cauchy-Schwarz inequality:

$$\leq \|\nabla(u-I_{h,p}u)\|_{L^2} \, \|\nabla(u-u_h)\|_{L^2}$$

Finally divide by the gradient norm of the error.

Estimating the gradient error

Situation now: We are comparing the error in the FE solution with the interpolation error:

$$\underbrace{\|\nabla(u-u_h)\|_{L^2}}_{\text{Finite element error}} \leq \underbrace{\|\nabla(u-I_{h,p}u)\|_{L^2}}_{\text{Piecewise polynomial approximation error}}$$

We call this a “best approximation estimate”: the finite element error is at least as good as the interpolation error.

For other equations, error estimation is more difficult. “Best approximation” would then look like this:

$$\underbrace{\|\nabla(u-u_h)\|_{L^2}}_{\text{Finite element error}} \leq \underbrace{C_{\text{equation}}}_{\text{Specifics of the equation}} \underbrace{\|\nabla(u-I_{h,p}u)\|_{L^2}}_{\text{Piecewise polynomial approximation error}}$$

Estimating the gradient error

Situation now: We are comparing the error in the FE solution with the interpolation error:

$$\underbrace{\|\nabla(u-u_h)\|_{L^2}}_{\text{Finite element error}} \leq \underbrace{\|\nabla(u-I_{h,p}u)\|_{L^2}}_{\text{Piecewise polynomial approximation error}}$$

Recall the interpolation error estimate (“Ingredient #2”) to obtain the final result:

$$\|\nabla(u-u_h)\|_{L^2} \leq \underbrace{\frac{C(p,\Omega)}{p!}}_{\text{constant}} \underbrace{h^p}_{\text{convergence!}} \underbrace{\|\nabla^{p+1}u\|_{L^2}}_{\text{property of the solution}}$$

Estimating the gradient error

For the Laplace equation, we have:

$$\|\nabla(u-u_h)\|_{L^2} \leq \underbrace{\frac{C(p,\Omega)}{p!}}_{\text{constant}} \underbrace{h^p}_{\text{convergence!}} \underbrace{\|\nabla^{p+1}u\|_{L^2}}_{\text{property of the solution}}$$

For many (but not all!) other equations, we get something like this:

$$\|\nabla(u-u_h)\|_{L^2} \leq \underbrace{C_{\text{equation}}}_{\text{specifics of the equations}} \underbrace{\frac{C(p,\Omega)}{p!}}_{\text{constant}} \underbrace{h^p}_{\text{optimal order convergence!}} \underbrace{\|\nabla^{p+1}u\|_{L^2}}_{\text{property of the solution}}$$

Estimating the L^2 error

For the Laplace equation, we have:

$$\|\nabla(u-u_h)\|_{L^2} \le \underbrace{\frac{C(p,\Omega)}{p!}}_{\text{constant}} \underbrace{h^p}_{\text{convergence!}} \underbrace{\|\nabla^{p+1}u\|_{L^2}}_{\text{property of the solution}}$$

With work (→ "Nitsche trick"), we can also estimate the L^2 error:

$$\|u-u_h\|_{L^2} \le \frac{C'(p,\Omega)}{p!} h^{p+1} \|\nabla^{p+1}u\|_{L^2}$$

Estimating the L^∞ error

For the Laplace equation, we have:

$$\|\nabla(u-u_h)\|_{L^2} \le \underbrace{\frac{C(p,\Omega)}{p!}}_{\text{constant}} \underbrace{h^p}_{\text{convergence!}} \underbrace{\|\nabla^{p+1}u\|_{L^2}}_{\text{property of the solution}}$$

With substantially more work, we also get:

$$\|u-u_h\|_{L^\infty} \le \frac{C''(p,\Omega)}{p!} h^{p+1} \|\nabla^{p+1}u\|_{L^2} \qquad \text{if } p>1$$

$$\|u-u_h\|_{L^\infty} \le \frac{C''(p,\Omega)}{p!} h^{p+1}\left(\log\frac{1}{h}\right) \|\nabla^{p+1}u\|_{L^2} \qquad \text{if } p=1$$

Summary

For the Laplace equation, estimating errors is based on two fundamental properties:

- Interpolation estimates
- Galerkin orthogonality

With these we get the following estimates that guarantee convergence and tell us how fast the error decreases with mesh refinement:

$$\|\nabla(u-u_h)\|_{L^2} \leq \frac{C(p,\Omega)}{p!} h^p \|\nabla^{p+1}u\|_{L^2}$$

$$\|u-u_h\|_{L^2} \leq \frac{C'(p,\Omega)}{p!} h^{p+1} \|\nabla^{p+1}u\|_{L^2}$$

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