MATH 546: Partial Differential Equations II

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Lectures: Engineering E 206, Mondays/Wednesdays/Fridays, 11-11:50am
Office hours: Wednesdays, 1-2pm; or by appointment.

Homework assignment 2 – due Friday 3/15/2019

Problem 1 (Solutions with singularities). In class we have discussed in detail the case where a solution to the Laplace equation has singularities (i.e., places where it is not as "regular" or "smooth" as one may wish for). In this case, the singularity was caused by the presence of a re-entrant corner of the domain. There are other reasons than just the geometry why this kind of thing may happen. One example is that the boundary conditions cause it.

Think, for example, of the case of a domain that is a half-circle: $\Omega = \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\| < 1, x_2 > 0\}$. Then ask that the solution satisfies Dirichlet boundary conditions $u = 0$ on the right part of the x_1 axis $(x \geq 0)$ and Neumann boundary conditions $\partial u/\partial n = -\partial u/\partial x_2 = h(x_1)$ on the left part of the x_1 axis $(x_1 < 0)$. As in the example in class, we will for the moment not want to prescribe any boundary conditions on the outer perimeter of the domain – we're just interested in figuring out whether there are solutions of the Laplace equation

$$-\Delta u = 0$$

that happen to be singular at the origin (where the two boundary conditions meet) if one allowed *anything* on the outer perimeter.

Can you construct such singular solutions if you allowed yourself free choice of $h(x_1)$ as long as h is a continuous function? In other words, can there be singular solutions to the Laplace equation even on domains that are convex (like the one chosen here) and if the right hand sides f, g, h are all smooth functions? Does any of this depend on the opening angle of the domain being $\theta = \pi$, or could you observe something similar for $\theta < \pi$? **(20 points)**

Problem 2 (Regularity of solutions of the Laplace equation). The solution of the Laplace equation mentioned above that we have derived in class for the case of a sector of a circular disk had the form

$$u(r, \phi) = r^{\pi/\theta} \sin(\pi\phi/\theta),$$

when expressed in polar coordinates. Here, θ is the opening angle of the sector $\Omega = \{(r, \phi) : 0 < r < R, 0 < \phi < \theta\} \subset \mathbb{R}^2$. Show the following two statements:

- This function (as a function of $(x, y) \in \mathbb{R}^2$) is in the function space $H^1(\Omega)$ for *any* opening angle θ, even those larger than π. For this, you need to show that both the function itself and its (weak) gradient are square integrable.
- This function is in $C^2(S)$ on any compact set $S < \Omega$. (This statement is a bit difficult to understand, though the wording is frequently used like this. To unpack it, first think about what it means for S to be compact. Then recall that the domains Ω on which we pose PDEs are always *open*. What does it then mean if S is a compact subset of Ω? Can you then conclude why this helps you with showing why $u \in C^2(S)$ even though we know that $u \notin C^2(\Omega)$?)

  As a side note, this is generally true for elliptic equations: If the right hand side is sufficiently smooth – e.g., $f \in C^0$ – then the solution will be $C^2(S)$ on any *compact* subset S, even though it may not be $C^2(\Omega)$. **(20 points)**

Problem 3 (Membership in $W^{k,p}$). Let's build some intuition. If one thinks of functions that have k derivatives, i.e., are in the set of functions usually denoted by $C^k(\Omega)$, then we think of them as quite smooth and definitely not singular. Is this true also for the functions in $W^{k,p}$ that have k *weak* derivatives whose pth power is integrable? Recall that

$$W^{k,p}(\Omega) = \left\{\varphi : \int |u|^p < \infty, \quad \int |\nabla u|^p < \infty, \quad \dots, \quad \int |\nabla^k u|^p < \infty\right\}.$$

To test whether a given function u is in $W^{k,p}$ for some k, p on *bounded* domains Ω, we really only have to check that the highest order term $\int |\nabla^k u|^p < \infty$ because if this is the case then all of the lower order derivatives also have finite integrals.

Now think about singular functions $u : B_1(0) \subset \mathbb{R}^d \to \mathbb{R} \cup \{\pm\infty\}$ of the form

$$u(x) = \frac{1}{\|x\|^s}$$

with $s > 0$. (The domain $\Omega = B_1(0)$ is the ball of radius 1 around the origin 0. For simplicity of calculations, use the Euclidean norm for $\|x\|$ – although the equivalence of norms on finite dimensional spaces implies that none of the results below actually depend on which norm $\|x\|$ you choose.)

For given values of the space dimension $d \geq 1$, the degree $k \geq 0$, and the exponent $1 \leq p \leq \infty$, state for which values s the function satisfies $u \in W^{k,p}(B_1(0))$.

The spaces $H^k = W^{k,2}$ have special importance in the theory of partial differential equations. Does the space $H^1 = W^{1,2}$ contain any singular functions with $s > 0$ for $d = 1$? For $d = 2$? For $d > 3$? How about the space $H^2 = W^{2,2}$? **(20 points)**

Problem 4 (Weak derivatives and membership in $W^{1,2}$). Not all discontinuous functions have a weak derivative. For example, the following function of one variable,

$$u(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x \geq 0, \end{cases}$$

does not have a weak derivative.

But how about the function $u : \mathbb{R}^2 \to \mathbb{R}$ of two arguments,

$$u(\mathbf{x}) = \sin(\arctan(x_2/x_1))$$

that is discontinuous at the origin? Can you guess a weak gradient for this function u (which is of course a two-dimensional vector field) and prove that it really is *the* weak gradient?

If so, what spaces $W^{1,p}(B_1(0))$ is u in if we restrict it to the unit ball in $\mathbb{R}^2$?

(Hint: Plot the function. Then think about whether there is possibly a coordinate system better suitable to the task.) **(20 points)**

Problem 5 (Weak formulation of the Stokes equations). In class, we have briefly shown how the weak formulation of the Stokes equations looks like. The Stokes equations can be derived as the steady state of the Navier-Stokes equations, under the additional assumption that the velocity is small because then the quadratic term $\mathbf{u} \cdot \nabla \mathbf{u}$ in the Navier-Stokes equations is negligible compared to all of the terms that are linear in $\mathbf{u}$. The model then looks like this (setting the viscosity to one):

$$-\Delta \mathbf{u}(\mathbf{x}) + \nabla p(\mathbf{x}) = \mathbf{f}(\mathbf{x}),$$
$$-\nabla \cdot \mathbf{u}(\mathbf{x}) = 0.$$

For boundary conditions, assume that you have the following (with $\Gamma_D \cup \Gamma_N = \partial\Omega$):

$$u = g \qquad \text{on } \Gamma_D \text{ (Dirichlet-type boundary conditions).}$$
$$\nabla u \cdot \mathbf{n} - p\mathbf{n} = h \qquad \text{on } \Gamma_N \text{ (Neumann-type boundary conditions).}$$

To convert this into the weak form, we want to think of the system of equations as one equation with vectors on the left and right:

$$\begin{pmatrix} -\Delta \mathbf{u} + \nabla p \\ -\nabla \cdot \mathbf{u} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ 0 \end{pmatrix}.$$

The solution is then the tuple $(\mathbf{u}, p)$ and we should multiply this with a test function of the form $(\mathbf{v}, q)$ and integrate over the domain:

$$\int_\Omega \begin{pmatrix} \mathbf{v} \\ q \end{pmatrix} \cdot \begin{pmatrix} -\Delta \mathbf{u} + \nabla p \\ -\nabla \cdot \mathbf{u} \end{pmatrix} = \int_\Omega \begin{pmatrix} \mathbf{v} \\ q \end{pmatrix} \cdot \begin{pmatrix} \mathbf{f} \\ 0 \end{pmatrix}.$$

Both left and right hand sides are now just numbers.

If you write all of the terms out, you get the following:

$$\int_\Omega \mathbf{v} \cdot (-\Delta \mathbf{u}) + \mathbf{v} \cdot \nabla p - q \nabla \cdot \mathbf{u} = \int_\Omega \mathbf{v} \cdot f.$$

From here, you will want to integrate by parts the term with the Laplace operator. Some terms that result from the integration by parts can be simplified using the boundary conditions. Go through the same checklist we have had in class and identify what properties you will need for the spaces U, P from which want to choose the solution $(\mathbf{u}, p)$. Do the same for V, Q for the test function $(\mathbf{v}, q)$.

Then consider what happens if you *also* integrate the term $\mathbf{v} \cdot \nabla p$ by parts and apply boundary conditions? Does this simplify things? Does it relax requirements on U, P, V, Q or does it make them more stringent?

(20 points)

Bonus problem (Solutions with singularities). The example in class and the one of Problem 1 shows that singularities in the solutions of the Laplace equation may be caused by the geometry (if the domain has corners and is not convex at these corners), or by a switch in the type of boundary condition (e.g., at points on the boundary where Dirichlet boundary conditions are switched to Neumann boundary conditions as you move along the boundary). In both cases, the singularities are located at the boundary.

But not all singularities must lie at the boundary. Can you construct a singular solution for the case where

$$-\Delta u(\mathbf{x}) = f(\mathbf{x})$$

where

$$f(\mathbf{x}) = \begin{cases} 1 & \text{if } x_1 \geq 0, x_2 \geq 0 \text{ (first quadrant)} \\ 0 & \text{if } x_1 < 0, x_2 \geq 0 \text{ (second quadrant)} \\ 1 & \text{if } x_1 < 0, x_2 < 0 \text{ (third quadrant)} \\ 0 & \text{if } x_1 \geq 0, x_2 < 0 \text{ (fourth quadrant).} \end{cases}$$

Here, the cause of the singularity is that there are multiple values for the right hand side coming together at a single point. In order to construct solutions, assume for a moment that the domain is the entirety of $\mathbb{R}^2$ so that we do not need to worry about boundary conditions. Then write the Laplace operator in polar coordinates and see whether you can piece a solution together for the four quadrants.

Can you show that the function is not in $C^2(B_1(0))$ but is in $H^1(B_1(0))$?

(If you find a function $\hat{u} : \mathbb{R}^2 \to \mathbb{R}$ that satisfies $-\Delta \hat{u} = f$ everywhere, then you can make this into a solution u of $-\Delta u = f$ on a bounded domain Ω if you require that $u|_{\partial\Omega} = g$ where $g = \hat{u}|_{\partial\Omega} = g$. This is of course not an approach that helps you find concrete solutions to problems where someone gives you a function g, but it shows that these kinds of singular solutions can happen if your boundary conditions happen to be that way.)

(10 bonus points)