Part 22

Integer programming problems

Terminology

Discrete optimization problems: If some or all variables can only take on certain values.

Example:

$$\begin{aligned} \min_x \quad & f(x) \\ \text{subject to} \quad & Ax \geq b \\ & x_1 \in \{1, 2.5, 3.75, 17\} \\ & x_2 \in \{2, 4, 6, 8\} \end{aligned}$$

Terminology

Integer optimization problems: If some or all variables can only take on *integer* values.

Example:

$$\begin{aligned} \min_x \quad & f(x) \\ \text{subject to} \quad & Ax \geq b \\ & x_1 \in \{1, 2, 3, 7\} \\ & x_2 \in \{2, 4, 6, 8\} \end{aligned}$$

Terminology

Note: We can typically convert discrete optimization problems into integer optimization problems.

Example:

$$\begin{aligned} &\min_x \quad f(x) \\ &\text{subject to} \quad Ax \geq b \\ &\qquad\qquad\quad x_1 \in \{1, 2.5, 3.75, 17\} \\ &\qquad\qquad\quad x_2 \in \{2, 4, 6, 8\} \end{aligned}$$

is equivalent to

$$\begin{aligned} &\min_{x,y} \quad f(x) \\ &\text{subject to} \quad Ax \geq b \\ &\qquad\qquad\quad x_2 \in \{2, 4, 6, 8\} \\ &\qquad\qquad\quad y_1, y_2, y_3, y_4 \in \{0, 1\} \\ &\qquad\qquad\quad y_1 + y_2 + y_3 + y_4 = 1 \\ &\qquad\qquad\quad x_1 = 1\, y_1 + 2.5\, y_2 + 3.75\, y_3 + 17\, y_4 \end{aligned}$$

Terminology

Integer linear program (ILP): If all variables can only take on *integer* values and the objective function is linear.

Example:

$$
\begin{aligned}
\min_{x_1, x_2} \quad & f(x) = c^T x \\
\text{subject to} \quad & Ax \geq b \\
& x_1 \in \{1, 2, 3, 7\} \\
& x_2 \in \{2, 4, 6, 8\}
\end{aligned}
$$

Terminology

Mixed integer linear program (MILP): If some but not all variables can only take on *integer* values and the objective function is linear.

Example:

$$\begin{aligned} \min_{x_1, x_2, x_3} \quad & f(x) = c^T x \\ \text{subject to} \quad & Ax \geq b \\ & x_1 \in \{1, 2, 3, 7\} \\ & x_2 \in \{2, 4, 6, 8\} \end{aligned}$$

Terminology

Mixed integer nonlinear program (MINP): If some but not all variables can only take on *integer* values and the objective function is nonlinear.

Example:

$$\begin{aligned} \min_{x_1, x_2, x_3} \quad & f(x) \\ \text{subject to} \quad & Ax \geq b \\ & x_1 \in \{1, 2, 3, 7\} \\ & x_2 \in \{2, 4, 6, 8\} \end{aligned}$$

Terminology

Zero-one linear program (ZOIP): If all variables can only take on *zero or one* values and the objective function is linear.

Example:

$$\min_{x,y,z} \; c^T x$$

$$\text{subject to} \quad Ax \geq b$$

$$y_1, y_2, y_3, y_4 \in \{0,1\}$$

$$z_1, z_2, z_3, z_4 \in \{0,1\}$$

$$y_1 + y_2 + y_3 + y_4 = 1$$

$$z_1 + z_2 + z_3 + z_4 = 1$$

$$x_1 = 1\, y_1 + 2.5\, y_2 + 3.75\, y_3 + 17\, y_4$$

$$x_2 = 2\, z_1 + 4\, z_2 + 6\, z_3 + 8\, z_4$$

Note: ILPs can typically be converted into ZOIPs.

Examples: The knapsack problem

Setting: Objects *1...N* with values c_i and weights w_i.

Goal: Pack maximal value subject to a weight constraint.

Formulation:

$$\begin{aligned} \max_x \quad & c^T x \\ \text{subject to} \quad & x_i \in \{0,1\} \\ & w^T x = \sum_i w_i x_i \le M \end{aligned}$$

Examples: The facility location problem

Setting: N potential facility locations to serve M clients. Fixed cost c_i to open a facility at i. Cost to serve client j from i is d_{ij}.

Goal: Select facility locations and assign clients to facilities while minimizing cost.

Formulation:

$$
\begin{aligned}
\min_{x_i,\, y_{ij}} \quad & \sum_i c_i x_i + \sum_i \sum_j d_{ij} y_{ij} \\
\text{subject to} \quad & x_i \in \{0,1\} \\
& y_{ij} \in \{0,1\} \\
& y_{ij} \leq x_i
\end{aligned}
$$

Note: The last constraint implies that we can only serve client j from location i if location i has actually been selected.

Examples: The traveling salesman problem (TSP)

Setting: N cities. Time to travel from i to j is t_{ij}.

Goal: Find a tour through all cities that takes the shortest amount of time.

Formulation:

$$
\begin{aligned}
\min_{x_{ij}} \quad & \sum_i \sum_j t_{ij} x_{ij} \\
\text{subject to} \quad & x_{ij} \in \{0,1\} \\
& \sum_j x_{ij} = 2 && \text{for every } i \\
& \sum_{i \in S} \sum_{j \notin S} x_{ij} \geq 2 && \text{for every } S \subset \{1 \ldots N\}, S \neq \emptyset
\end{aligned}
$$

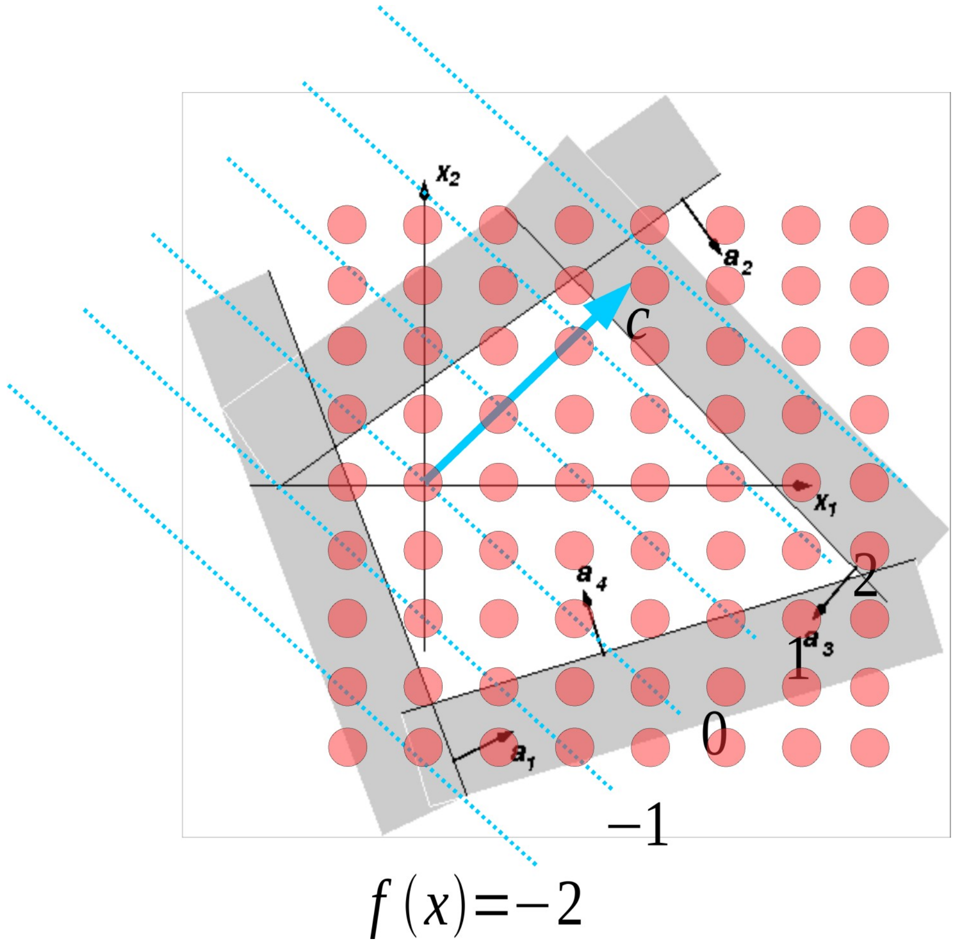
Note: The second constraint implies that each city has exactly two selected edges. The third that each proper subset of cities must have at least two edges selected to the rest of the cities.

The geometry of ILP problems

Consider the problem

$$\begin{array}{ll} \min_{x_i} & c^T x \\ \text{subject to} & x_i \in \mathbb{N} \\ & Ax \geq b \end{array}$$

Feasible points form a lattice inside the polyhedron described by the linear inequalities:



In general, we will have to try a substantial fraction of points to find the optimum!

Solution strategies: Branch and bound

Consider the problem

$$\begin{aligned} \min_{x_i} \quad & c^T x \\ \text{subject to} \quad & x_i \in \{0,1\} \\ & Ax \geq b \end{aligned}$$

Think of this as a decision tree that we can enumerate:

- Pick $x_1 = 0$
  - Are constraints satisfiable? If yes, pick $x_2 = 0$
    - Are constraints satisfiable? If yes, pick $x_3 = 0$
      - …
  - Choose $x_2 = 1$
    - Are constraints satisfiable? If yes, pick $x_3 = 0$
      - …

We visit every node of the tree. Keep track of the best one seen so far.

Solution strategies: Branch and bound

Consider the problem

$$\begin{aligned} \min_{x_i} \quad & c^T x \\ \text{subject to} \quad & x_i \in \{0,1\} \\ & Ax \geq b \end{aligned}$$

Think of this as a decision tree that we can enumerate.

Improvement 1:

If the cost so far exceeds the cost of the best path encountered, we need not continue searching in a sub-tree.

Solution strategies: Branch and bound

Consider the problem

$$\begin{aligned} \min_{x_i} \quad & c^T x \\ \text{subject to} \quad & x_i \in \{0,1\} \\ & Ax \geq b \end{aligned}$$

Think of this as a decision tree that we can enumerate.

Improvement 2:

If the cost so far *plus a lower bound for the cost of the rest of the path* exceeds the cost of the best path encountered, we need not continue searching in a sub-tree.

Solution strategies: Branch and bound

Think this through for the Traveling Salesman Problem:

Consider four cities in a square around a desert, with travel times

- 1 – 2: 1 hour
- 2 – 3: 1 hour
- 3 – 4: 1 hour
- 1 – 4: 1 hour
- 1 – 3: 4 hours
- 1 – 4: 3 hours

The shortest tour is clearly 4 hours.