MATH 545: Partial Differential Equations I

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Lectures: Engineering E 206, Mondays/Wednesdays/Fridays, 10-10:50am
Office hours: Wednesdays, 1-2pm; or by appointment.

Homework assignment 2 – due Friday 9/28/2018

Problem 1 (Divergence theorem). If a function $u(x, y, z)$ satisfies $-\Delta u = 0$ at every point of a domain $\Omega \subset \mathbb{R}^3$, show that for any bounded subset $\omega \subset \Omega$

$$\int_{\partial\omega} \vec{n} \cdot \nabla u \, ds = 0$$

holds, where $\partial\omega$ is the surface (boundary) of ω. **(10 points)**

Problem 2 (Not quite the heat equation). Flea migration works a bit like heat transport: they hop around randomly, and after a while they're everywhere. In much the same way as for the heat equation, derive an equation and boundary conditions for the density of fleas $u(\vec{x}, t)$ under the following premises:

- The space where fleas hop around is a two-dimensional area Ω ("the room").
- Fleas hop around randomly; consequently if the density of fleas to the right of a line is n times higher than to the left, n times as many fleas will cross it from the right to the left than the other way around (think of what such reasoning meant for the heat *flux* in relation to the temperature).
- The room has walls through which fleas neither leave not enter.

In the process of deriving your equations for the differential equation/boundary conditions/initial conditions, identify the coefficients analogous to heat conductivity, density, etc in the equation you get. State "physical" units for each quantity that appears in your equation and make sure that the equation is dimensionally correct.

Intuitively, since fleas neither leave nor enter the room, their total number must be constant. Can you derive this from the equations? **(25 points)**

Problem 3 (Eigenfunctions of $\frac{\partial^2}{\partial x^2}$). As part of solving the heat equation for one space dimension, we had to find the solutions of the equations

$$-\frac{\partial^2 \phi(x)}{\partial x^2} = \lambda \phi(x), \qquad \phi(0) = 0, \qquad \phi(L) = 0.$$

The (non-trivial) solutions were $\phi_n(x) = \sin(n\pi x/L)$, for $n = 1, 2, \ldots$ Repeat this exercise by finding the solutions of the eigenproblem

$$-\frac{\partial^2 \phi(x)}{\partial x^2} = \lambda \phi(x), \qquad \phi(0) = 0, \qquad \frac{\partial \phi}{\partial x}(L) = 0,$$

where only the boundary condition at the right has been changed. What can you infer about the sign of λ now, and for which values of λ can you find solutions $\phi(x)$? **(15 points)**

Problem 4 (Solutions of the heat equation). In class, we have derived that the general solution of the one-dimensional heat equation

$$\frac{\partial u(x,t)}{\partial t} - k\frac{\partial^2 u(x,t)}{\partial x^2} = 0, \qquad u(0,t) = 0, \qquad u(L,t) = 0$$

has the form

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-\frac{kn^2\pi^2}{L^2}t} \sin\left(\frac{n\pi}{L}x\right).$$

For each of the following initial conditions $u_0(x)$, find the values of the coefficients B_n so that $u(x,0) = u_0(x)$, put these B_n into the general form above, and see if you can simplify the expression if possible:

(a) $u_0(x) = 6\sin\left(\frac{9\pi}{L}x\right)$

(b) $u_0(x) = 3\sin\left(\frac{\pi}{L}x\right) - \sin\left(\frac{3\pi}{L}x\right)$

(c) $u_0(x) = 2\cos\left(\frac{2\pi}{L}x\right)$

(d) $u_0(x) = \begin{cases} 1 & \text{if } x < L/2 \\ 2 & \text{if } x \geq L/2. \end{cases}$

You may remark that not all of these actually satisfy the boundary conditions – but this is not physically wrong: at $t = 0$, you just stick the end of an initially hot piece of metal into a cold bath.

(35 points)

Problem 5 (Visualizing solutions of the heat equation). For each of the solutions of the four sub-problems of the previous problem, visualize the solution $u(x,t)$ if you choose $L = 1$, $k = 1$ on the interval $0 \leq x \leq L$ and $0 \leq t \leq 0.5$.

Interpret what you see and how that matches your physical interpretation if you imagine that $u(x,t)$ represents the temperature of a one-dimensional rod and how it changes with time.

(15 points)