MATH 437: Principles of Numerical Analysis

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Homework assignment 4 – due Thursday 9/26/2013

Problem 1 (Norms on $\mathbb{R}^n$). A functional $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}_0^+$ is a norm if it satisfies the following three conditions:

- $\|x\| = 0$ if and only if $x = 0$ (positivity);
- $\|\lambda x\| = |\lambda|\|x\|$ for all $\lambda \in \mathbb{R}$ and all vectors $x \in \mathbb{R}^n$ (linearity);
- $\|x + y\| \le \|x\| + \|y\|$ for all vectors $x, y \in \mathbb{R}^n$ (triangle inequality).

Determine which of the following are norms on $\mathbb{R}^n$ by proving or disproving that they satisfy the three conditions above:

a) $\max\{|x_2|, |x_3|, |x_4|, \dots, |x_n|\}$

b) $\sum_{i=1}^n |x_i|^3$

c) $\left(\sum_{i=1}^n |x_i|^{1/2}\right)^2$ (this corresponds to the $l_{1/2}$ norm of the l_p norm family)

d) $\max\{|x_1 - x_2|, |x_1 + x_2|, |x_3|, |x_4|, \dots, |x_n|\}$

e) $\sum_{i=1}^n 2^{-i}|x_i|$ (this looks like a variant of the l_1 norm where each entry is weighted with the positive number 2^{-i}).

Hint: If you suspect that something may not be a norm, it is often easiest to find an example of a vector x (or of vectors x, y) for which one of the conditions for being a norm is violated. **(5 points)**

Problem 2 (Norms on $\mathbb{R}^n$). Various different norms, such as the ones in Problem 1, are not introduced for intellectual pleasure, but because they are useful in many situations. For example, it is often much simpler to show something about a particular matrix in the l_1 norm,

$$\|A\|_1 = \max_j \left(\sum_i |a_{ij}| \right),$$

because it is easy to compute. On the other hand, the l_2 norm,

$$\|A\|_2 = \max_i \sqrt{\lambda_i(A^T A)},$$

involves the eigenvalues λ_i of the matrix $A^T A$ and is therefore not easy to compute. In particular, there is no simple relationship between the eigenvalues of $A^T A$ (or A for that matter) and the matrix entries of A.

So what if I am interested in a certain property that involves the l_p norm of a vector (for a given $1 \leq p \leq \infty$), but I can only prove it for the l_q norm with q different from p? It turns out that this doesn't matter in many cases: for (finite-dimensional) vector spaces, all norms are equivalent, i.e. if $\|\cdot\|_p$ and $\|\cdot\|_q$ are norms, then there are constants $c, C > 0$ such that

$$c\|v\|_q \leq \|v\|_p \leq C\|v\|_q.$$

For example, in class it was shown that the Jacobi iteration converges in the l_∞ norm as $\|x_k - x\|_\infty \leq \delta^k \|x_0 - x\|_\infty$. If we can show above inequality, then we also know that $\|x_k - x\|_1 \leq C\delta^k \|x_0 - x\|_\infty \leq \frac{C}{c}\delta^k \|x_0 - x\|_1$ for some c, C.

a) Prove that there are indeed constants c, C such that

$$c\|v\|_\infty \leq \|v\|_1 \leq C\|v\|_\infty.$$

where

$$\|v\|_1 = \sum_i |v_i|,$$
$$\|v\|_\infty = \max_i |v_i|,$$

and where v is an n-dimensional vector in $\mathbb{R}^n$.

b) For vectors v_1, v_2 with $\|v_1\|_1 \leq \|v_2\|_1$, does the result of part a) imply that $\|v_1\|_\infty \leq \|v_2\|_\infty$? If not, give an example of vectors for which this does not follow.

(4 points)

Problem 3 (Convergence of Jacobi iteration). Let A, b be the 100×100 matrix and 100-dimensional vector defined by

$$A_{ij} = \begin{cases} 2.01 & \text{if } i = j, \\ -1 & \text{if } i = j \pm 1, \\ 0 & \text{otherwise,} \end{cases} \qquad b_i = \frac{1}{100} \sin\left(\frac{2\pi i}{50}\right).$$

Apply Jacobi's method to solving $Ax = b$. Write a program that implements the Jacobi method and start with a vector x_0 with randomly chosen elements in the range $0 \leq (x_0)_i \leq 1$ (i.e. with elements generated from what the `rand()` function or a similar replacement returns).

Run 200 Jacobi iterations and plot the values of $(x_k)_i$ against i for every few iterations, for example $k = 0, 2, 5, 10, 20, 50, 100, 200$. What do you observe?

(5 points)