MATH 601: Quiz 8 (10/31/2012)

Name: UIN:

Problem 1 (5 points): Consider the following set of vectors in $V = \mathbb{R}^3$:

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix}, \quad u_4 = \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix}.$$

Answer these questions:

- Are these vectors linearly independent? If not, why not?

  Answer: No. The short answer would be that because the dimension of V is three, there can not be four linearly independent vectors in any set of vectors from V. The long answer is that we can show that we can express at least one of these vectors as a linear combination of the others. For example, $u_4 = \frac{5}{4}u_3 - \begin{pmatrix} 0 \\ 15/4 \\ 0 \end{pmatrix} = \frac{5}{4}u_3 - \frac{15}{8}(u_2 - u_1)$.

- Do these vectors span V? If not, why not?

  Answer: Yes, one can indeed write every vector in $\mathbb{R}^3$ as a linear combination of these four vectors.

- Are these vectors a basis of V? If not, why not?

  Answer: No. A basis is defined as a set of spanning vectors (which these four vectors are) that has exactly as many elements as the dimension of this space (this is the part that is violated here).

- If the answer to the previous question was no, provide a set of vectors that form a basis of V.

  Answer: The first three of the vectors above will do. As do various other combinations (maybe even all combinations of three of the above, but that would need to be verified). A different, simpler basis would be the set of vectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

- What is the dimension of V?

  Answer: 3. The three vectors given in the previous part of the problem are linearly independent as is easy to verify, and they span the entire space. Thus, the dimension of the space is three.

(see backside)

Problem 2 (5 points): Consider the following set of vectors in the vector space $V = \mathbb{S}^{2\times 2}$ of *symmetric* 2×2 matrices:

$$u_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix}.$$

Answer these questions:

- Are these vectors linearly independent? If not, why not?

  Answer: Yes. None of these matrices can be written as a linear combination of the others.

- Do these vectors span V? If not, why not?

  Answer: Yes. In some answers to this question the argument was made that one can not write the matrix $M = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ as a linear combination of the three given matrices. However, M is not an element of the space $\mathbb{S}^{2\times 2}$ (because it is not symmetric) and so there is no need to be able to write it as a linear combination of the three matrices.

  In other words, the three vectors span the space $V = \mathbb{S}^{2\times 2}$. They do not span the space $\mathbb{R}^{2\times 2}$ of all 2×2 matrices, symmetric or not, but this was not the question.

- Are these vectors a basis of V? If not, why not?

  Answer: Yes, they do form a basis: they span the space and this set of three matrices has the least possible cardinality of sets that span the space (namely, three).

- If the answer to the previous question was no, provide a set of vectors that form a basis of V.

  Answer: The answer was *yes*, and so no answer is necessary here. That said, if you want, another basis for V would have been the set of the following matrices:

$$v_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \frac{1}{2}(u_1 + u_2), \qquad v_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2}(u_1 - u_2), \qquad v_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -u_3 + (u_1 + u_2).$$

  Some answers gave the following set of basis vectors:

$$v_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad v_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad v_4 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

  This is incorrect, however: the last two matrices are not in $\mathbb{S}^{2\times 2}$ since they are not symmetric and so they can not be part of a basis of this space.

- What is the dimension of V?

  Answer: 3. The dimension of the space $\mathbb{R}^{2\times 2}$ would be 4 but one can only find sets of three linearly independent *symmetric* matrices.