MATH 601: QUIZ 11 (11/28/2012)

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Problem 1 (4 points): Consider the function $f(z) = \bar{z}$ where the complex conjugate $\bar{z}$ of a number $z = x + iy \in \mathbb{C}$ is defined as $\bar{z} = x - iy$. Answer the following questions:

- Is $f(z)$ linear? If not, why not?

  Answer: No. To be linear, f would have to satisfy the relationship $f(\alpha z) = \alpha z$ for every argument $z \in \mathbb{C}$ and every multiplier $\alpha \in \mathbb{C}$. However, we can easily find values α, z for which this relationship is not true. For example, with $z = i, \alpha = i$ we have that $f(\alpha z) = \overline{i^2} = \overline{-1} = -1$ but $\alpha f(z) = i(\bar{i}) = i(-i) = 1 \neq f(\alpha z)$.

- Is $f(z)$ differentiable as a complex function? If not, why not?

  Answer: No, but this has nothing to do with the fact that $f(z)$ is not linear: many non-nonlinear functions (e.g., $g(z)z^2$) *are* differentiable.

  For $f(z)$ to be differentiable we have to consider the Cauchy-Riemann equations. To this end, let us write $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ where u, v are the real and imaginary parts of f. For the given f, we thus have $u(x, y) = x, v(x, y) = -y$. The Cauchy-Riemann equations require that

  $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

  Here, we have

  $$\frac{\partial u}{\partial x} = 1, \qquad \frac{\partial v}{\partial y} = -1, \qquad \frac{\partial u}{\partial y} = 0, \qquad \frac{\partial v}{\partial x} = 0.$$

  In other words, the first Cauchy-Riemann equation is not satisfied (but the second one is) and so the function can not be differentiable.

  To see why linearity is not a requirement for differentiability, it is probably a good exercise if you tried this same line of reasoning on the function $g(z) = z^2$ mentioned above and see for yourself that in that case, the Cauchy-Riemann equations are indeed satisfied.

- Is $f(z)$ analytic? If not, why not?

  Answer: No. A function needs to at least be differentiable to be analytic.

(see backside)

Problem 2 (4 points): Consider the function $f(z) = z^2$ and the curve C given by all the points along the circle of unit distance from the origin. Then answer the following questions:

- At which points of the complex plane is $f(z)$ analytic?

  Answer: It is easy to verify that the function $f(z)$ is analytic *everywhere*, i.e., for all points $z \in \mathbb{C}$.

- Is the curve C closed? If so, describe the region $\Omega \subset \mathbb{C}$ enclosed by C. Is this region simply connected or multiply connected?

  Answer: Yes, C is closed (it is a circle). The region enclosed is then the unit disk, i.e., all points with distance less than one from the origin. The unit disk is simply connected.

- Compute the integral $\int_C f(z)\, \mathrm{d}z$.

  Answer: Because the curve C is the complete boundary of a region in which f is analytic, we can use Cauchy's theorem to state that $\int_C f(z)\, \mathrm{d}z = 0$. One could of course also compute this directly and would, in that case, find that the result is still zero.

Problem 3 (2 points): For each of the following subsets of $\mathbb{C}$, sketch a picture of how this subset looks and state whether it is simply or multiply connected.

- $\Omega = \{\text{all points of the form } a + bi \text{ where } a, b \in \mathbb{N}\}$.

  Answer: This subset consists of all points in the complex plane where both real and imaginar components are positive integers (i.e., natural numbers). Thus, it is a *lattice* of individual points in the interior of the first quadrant that we can enumerate if we want. This subset is not simply connected because to draw any curve that connects, for example, $z_1 = 1 + i, z_2 = 2 + i \in \Omega$ we would have to cross into parts of the complex plane that is not a subset of Ω. The condition for a region to be simply connected required us that there exists at least one such curve completely enclosed in Ω that connects z_1, z_2.

- $\Omega = \{z \in \mathbb{C} : |z - i| \leq 1 \text{ or } |z + i| \leq 1\}$.

  Answer: This region is the union of disks (including their boundaries) of radius one centered at $\pm i$. They touch at $z = 0$. Since $z = 0$ is inside Ω we can find curves completely enclosed in Ω that connect points in the two disks and, consequently, Ω is simply connected.

- $\Omega = \{z \in \mathbb{C} : |z - i| < 1 \text{ or } |z + i| < 1\}$.

  Answer: This slight variant of the previous question is tricky: Ω is composed of two disks of all points that are *less than* one unit away from the centers $\pm i$. This, in particular, does not include the point $z = 0$. In other words, both disks come arbitrarily close to $z = 0$ but there is no point that is part of both. Consequently, one can not find a curve entirely contains in Ω that connects points from the two disks. The region is therefore no simply connected.