MATH 417: Numerical Analysis

Instructor: Prof. Wolfgang Bangerth
bangerth@math.tamu.edu,

Teaching Assistants: Dukjin Nam
dnam@math.tamu.edu

Homework assignment 5 – due 3/8/2007

Problem 1 (LU decomposition). Write a program that implements the LU decomposition algorithm for general $n \times n$ matrices and outputs the L and U factors. Apply it to the matrix of the linear system

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}.$$

In a second step, implement the backward and forward substitution solves with the upper and lower triangular factors L and U for any given vector. Apply it to the right hand side above and verify that your solution is correct.

(6 points)

Problem 2 (Norms on $\mathbb{R}^n$). In the analysis of iterative solution methods for linear systems, we will come across different vector norms. A functional $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ is called a *norm* if it satisfies the following three conditions:

(a) $\|x\| \geq 0$ for all vectors $x \in \mathbb{R}^n$ and $\|x\| = 0$ if and only if $x = 0$ (positive definiteness);

(b) $\|\lambda x\| = |\lambda|\|x\|$ for all $\lambda \in \mathbb{R}$ and all vectors $x \in \mathbb{R}^n$ (scalability);

(c) $\|x + y\| \leq \|x\| + \|y\|$ for all vectors $x, y \in \mathbb{R}^n$ (triangle inequality).

Determine which of the following are norms on $\mathbb{R}^n$ by proving or disproving that they satisfy the three conditions above:

a) $\max_{1 \leq i \leq n} |x_i|$

b) $\max_{2 \leq i \leq n} |x_i|$

c) $\sum_{i=1}^n |x_i|^3$

d) $\left(\sum_{i=1}^n |x_i|^{1/2}\right)^2$

e) $\max\{|x_1 - x_2|, |x_1 + x_2|, |x_3|, |x_4|, \ldots, |x_n|\}$

f) $\sum_{i=1}^n 2^{-i}|x_i|$ **(6 points)**

(please turn over)

Problem 3 (Vector and matrix norms). Compute the norms $\|.\|_\infty$ and $\|.\|_2$ for

$$\begin{aligned} x_1 =& (3, -4, 0, 1.5)^T \\ x_2 =& (2, 1, -3, 4)^T \\ x_3 =& (\sin k, \cos k, 2^k)^T \quad \text{for } k \in \mathbb{N} \\ x_4 =& \left(\tfrac{4}{k+1}, \tfrac{2}{k^2}, k^2 e^{-k}\right)^T \quad \text{for } k \in \mathbb{N} \end{aligned}$$

Compute the row-sum norm $\|.\|_\infty$ of the following matrices:

$$A = \begin{pmatrix} 10 & 15 \\ 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 10 & 0 \\ 15 & 1 \end{pmatrix}$$

$$C = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \qquad D = \begin{pmatrix} 4 & -1 & 7 \\ -1 & 4 & 0 \\ -7 & 0 & 4 \end{pmatrix}$$

(4 points)

Problem 4 (Equivalence of norms on $\mathbb{R}^n$). In class, we proved the equivalence of the norms $\|.\|_\infty$ and $\|.\|_2$. Here now, prove the same for $\|.\|_\infty$ and $\|.\|_1$, where

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|.$$

a) Prove that there are indeed constants c, C such that

$$c\|v\|_\infty \leq \|v\|_1 \leq C\|v\|_\infty.$$

where

$$\begin{aligned} \|v\|_1 &= \sum_i |v_i|, \\ \|v\|_\infty &= \max_i |v_i|, \end{aligned}$$

and where v is an n-dimensional vector in $\mathbb{R}^n$.

b) For vectors v_1, v_2 with $\|v_1\|_1 \leq \|v_2\|_1$, does the result of part a) imply that $\|v_1\|_\infty \leq \|v_2\|_\infty$? If not, give an example of vectors for which this does not follow. **(4 points)**