MATH 412: Theory of Partial Differential Equations

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Homework assignment 1 – due Thursday 9/6/2007

Problem 1 (Bivariate analysis). Here is a picture of the large radio telescope in Arecibo, Puerto Rico:

Impose a coordinate system with the origin at the center of the dish and such that the positive x-axis runs from the origin in the direction of the tower in front. Let Ω be the domain in x-y-space occupied by the dish. Let $H(x, y)$ be the height of the telescope's surface above the level defined by the circular rim (the surface is of course below the rim, so $H(x, y) \leq 0$).

a) Plot the coordinate system (i.e. x- and y-axes) into the picture. Indicate $H(0, 0)$.

b) Describe in words the meaning of the following quantities defined on the entire domain and state the sign of the quantities on the second line:

$$\frac{\partial H(x,y)}{\partial x} \qquad \frac{\partial H(x,y)}{\partial y} \qquad \nabla H(x,y)$$

$$\frac{\partial^2 H(x,y)}{\partial x^2} \qquad \frac{\partial^2 H(x,y)}{\partial y^2} \qquad \Delta H(x,y)$$

$$\int_\Omega H(x,y)\,dx\,dy \qquad \int_{-R}^{R} H(x,0)\,dx \qquad \nabla H(0,0)$$

c) Describe in words the meaning of the following quantities defined on the boundary of the domain and state the sign of quantities where possible:

$$\mathbf{n} \qquad \frac{\partial H(x,y)}{\partial n} = \mathbf{n} \cdot \nabla \partial H(x,y) \qquad \frac{\partial^2 H(x,y)}{\partial n^2}$$

$$\int_{\partial\Omega} H(x,y)\,ds \qquad \int_{\partial\Omega} \frac{\partial H(x,y)}{\partial n}\,ds$$

(5 points)

Problem 2 (Integration by parts 1). Calculate the following integrals using integration by parts:

a) $\int_0^\pi x \sin x \; dx$

b) $\int_0^1 x e^x \; dx$

c) $\int_0^1 x^3 e^x \; dx$

(3 points)

Problem 3 (Integration by parts 2). Using one of the remarkable identities linking the fundamental constants e and π, namely $\int_{-\infty}^{\infty} e^{-x^2} \; dx = \sqrt{\pi}$, you show that the second moment of the Gaussian bell curve satisfies

$$\int_{-\infty}^{\infty} x^2 e^{-x^2} \; dx = \frac{1}{2}\sqrt{\pi}.$$

(Hint: in the integration by parts formula $\int u'v \; dx = -\int uv' \; dx$+boundary terms, where $u'(x)v(x) = x^2 e^{-x^2}$, you may want to identify $u'(x) = -2xe^{-x^2}, v(x) = -\frac{1}{2}x$ and proceed from there.)

Also show that the first moment of the Gaussian bell curve is zero, i.e.

$$\int_{-\infty}^{\infty} x e^{-x^2} \; dx = 0.$$

(3 points)

Problem 4 (Integration by parts 3). Let $f(x,y) = x^2 + y^2$ and $g(x,y) = \sin(xy)$. State which of the following statements is true and why or why not:

a) $\int_{-\pi}^{\pi} f(x,0)g(x,0)\,dx = 0$

b) for every y there holds

$$\int_{-1}^{1} \frac{\partial f(x,y)}{\partial x} g(x,y)\,dx$$
$$= -\int_{-1}^{1} \frac{\partial g(x,y)}{\partial x} f(x,y)\,dx + f(1,y)g(1,y) - f(-1,y)g(-1,y)$$

c) for every y there holds (note the signs)

$$\int_{-1}^{1} \frac{\partial f(x,y)}{\partial x} g(x,y)\,dx$$
$$= +\int_{-1}^{1} \frac{\partial g(x,y)}{\partial x} f(x,y)\,dx - f(1,y)g(1,y) + f(-1,y)g(-1,y)$$

d) for every x there holds

$$\int_{-1}^{1} \frac{\partial f(x,y)}{\partial x} g(x,y)\,dy$$
$$= -\int_{-1}^{1} \frac{\partial g(x,y)}{\partial x} f(x,y)\,dy + f(x,1)g(x,1) - f(x,-1)g(x,-1)$$

(4 points)

Problem 5 (Divergence theorem). For the simple case of the unit square $\Omega = [0,1]^2$, show that the divergence theorem

$$\int_{\Omega} \operatorname{div} \mathbf{u}\,dx\,dy = \int_{\partial\Omega} \mathbf{n} \cdot \mathbf{u}\,dl$$

holds for all sufficiently smooth vector fields $\mathbf{u}$. Hint: Use that the integral over Ω is really an integral over $0 \le x, y \le 1$ and that in the surface integral on the right you can express the normal vector explicitly on each part of the boundary.
(3 points)