MATH 412: Theory of Partial Differential Equations

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Homework assignment 11 – due Thursday 11/30/2006

Problem 1 (Eigenfunction expansion of the wave equation). This problem is similar to Problem 4 on last week's homework. Consider the wave equation

$$\begin{aligned}\frac{\partial^2 u(x,t)}{\partial t^2} - c^2 \frac{\partial^2 u(x,t)}{\partial x^2} &= q(x,t),\\ u(0,t) &= 0,\\ u(L,t) &= 0,\\ u(x,0) &= f(x),\\ \frac{\partial}{\partial t}u(x,0) &= g(x).\end{aligned}$$

As for the heat equation, the solution can be written as

$$u(x,t) = \sum_{n=1}^{\infty} A_n(t) \sin\frac{n\pi x}{L}.$$

Again, the coefficients $A_n(t)$ have to satisfy a different ordinary differential equation than the coefficients of the wave equation. Go back to your notes to see how the ODE for $a_n(t)$ was derived for the heat equation and adjust this process to the present equation. Derive and explain which ODE $A_n(t)$ has to satisfy here, and how we can derive initial conditions $A_n(0)$ and $\frac{dA_n}{dt}(0)$. Also explain why we need two initial conditions for A_n.

(3 points)

Problem 2 (Solution of the wave equation). The solution to Problem 1 is that the solution to the wave equation in 1-d, assuming a length $L = 1$, is

$$u(x,t) = \sum_{n=1}^{\infty} A_n(t) \sin(n\pi x),$$

and that the coefficients $A_n(t)$ are

$$A_n(t) = \alpha_n \cos(\omega_n t) + \beta_n \sin(\omega_n t) + \int_0^t q_n(t) \frac{\sin(\omega_n t - \omega_n \tau)}{\omega_n}\, d\tau,$$

where $\omega_n = cn\pi$ and

$$\alpha_n = \frac{\int_0^1 f(x)\sin(n\pi x)\ dx}{\int_0^1 \left(\sin(n\pi x)\right)^2\ dx}, \qquad \beta_n = \frac{1}{\omega_n}\frac{\int_0^1 g(x)\sin(n\pi x)\ dx}{\int_0^1 \left(\sin(n\pi x)\right)^2\ dx},$$

$$q_n(t) = \frac{\int_0^1 q(x,t)\sin(n\pi x)\ dx}{\int_0^1 \left(\sin(n\pi x)\right)^2\ dx}.$$

Compute the terms $\alpha_n, \beta_n, q_n(t)$ for the following set of initial conditions and source terms:

$$q(x,t) = 0, \qquad f(x) = \begin{cases} x & \text{for } x < \frac{1}{2} \\ 1-x & \text{for } x \geq \frac{1}{2} \end{cases}, \qquad g(x) = 0.$$

This corresponds to a string that is picked exactly in the middle and forms a straight line from the two ends (where it is clamped to zero elevation) to the center point.

With these coefficients, state the complete form of the solution to the problem. (Hint: The solution is $\alpha_n = 0$ for even n, $\alpha_n = \frac{4}{n^2\pi^2}(-1)^{\frac{n-1}{2}}$ for odd n, $\beta_n = q_n(t) = 0$. However, you should explain why.) **(5 points)**

Problem 3 (A numerical demonstration of the wave equation). In Problem 2, you have derived the solution of the wave equation for a particular set of initial and source conditions. Let the wave speed $c = 1$. Then we had that

$$u(x,t) = \sum_{n=1}^{\infty} A_n(t)\sin(n\pi x),$$

where A_n satisfies the equations given in the previous problem. Let us approximate this solution by not taking infinitely many terms, but truncating the sum at a finite number, let's say 20:

$$\tilde{u}(x,t) = \sum_{n=1}^{20} A_n(t)\sin(n\pi x).$$

For given values of x and t, you now have all the information to compute $\tilde{u}(x,t)$. Write a computer program that does exactly this. Generate plots of $\tilde{u}(x,t)$ at $t = 0, \frac{1}{10}, \frac{2}{10}, \frac{3}{10}, \dots, \frac{30}{10}$, for example by sampling x in intervals of 0.01. **(4 points)**

Bonus question: 3 extra points for everyone who manages to produce a movie out of this function $\tilde{u}(x,t)$ showing how this string vibrates. 1 further extra point if you take not only the first 20, but the first 100 terms of the infinite sum into account. **(up to 4 points)**

Happy Thanksgiving!