Western Number Theory Problems, 16 & 19 Dec 2002

Edited by Gerry Myerson

for distribution prior to 2003 (Asilomar) meeting

Summary of earlier meetings & problem sets with old (pre 1984) & new numbering.

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2000 San Diego	000:01–000:15	2001 Asilomar	001:01–001:23
2002 San Francisco (current set) 002:01–002:24			

[With comments on 94:23, 001:13, 001:16, and 001:22]

COMMENTS ON ANY PROBLEM WELCOME AT ANY TIME

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94:23 (Zachary Franco) For which t does $t^a + t^b + 1 = n^2$ have a solution in positive integers a, b, n ? It's clear that if $(3|(t-1)) = -1$, then no solutions exist.

Remarks: It was conjectured that for $t = 2$ the only solutions were $(a, b, n) = (5, 4, 7)$ and $(2k, k+1, 2^k+1)$ for $k = 1, 2, 3, \ldots$, but Robert Styer, in a 94-12-30 email, gave $(9, 4, 23)$, and said that there were no other sporadic solutions with $n \le 2^{20}$.

The problem was emailed to Reese Scott and Benne de Weger. Reese Scott shows that if a is even, only the infinite family of solutions exists. If a is odd and $a \le 3b-3$, then there are only the two sporadic solutions. If a is odd and $a > 3b-3$, he hadn't solved it at the time of writing, but if a solution exists, $a > 40$.

de Weger notes that the following are relevant:

Frits Beukers, On the generalized Ramanujan-Nagell equation I, II, *Acta Arith.*, **38**(1980/81) 389–410, **39**(1981) 113–123; *MR* **83a**:10028a,b.

namely, for the R-N equation $2^a + D = n^2$, Beukers proves $a \le 435 + 10\ln|D|/\ln 2$, so for the present equation $a \le 435 + 10b$.

Reese Scott said that the methods he used for $t = 2$ apply for t prime and, for prime p, $p^a + p^b + 1 = n^2$ has no solutions unless $p \equiv 7 \bmod 8$, and there are no solutions unless a is odd and b is even, and no solutions if $a \le 3b$. Hence he shows that $n > 2 \cdot 10^8$.

Remark: (new) Gary Walsh notes that Florian Luca has a fairly general result on this problem. A preliminary version, "The diophantine equation $x^2 = p^a \pm p^b + 1$," is available. Gary writes, "The main point is that the hypergeometric method (used by Beukers, Bennett-Bauer, and others) is not always necessary, and in the case that t is prime, a fairly simple argument using only basic algebraic number theory suffices." See also **002:21**.

001:13 (Chris Smyth) What is the greatest degree of an algebraic number whose conjugates span a 4-dimensional vector space over the rationals? It is known that the degree cannot exceed 1152, and an example of degree 384 is known.

Solution: Noam Elkies writes, "The upper bound of 1152 is attained. Let G be the 1152-element subgroup of $\mathrm{GL}_4(\mathbf{Q})$ generated by the signed coordinate permutations and the scaled Hadamard matrix $[1,1,1,1;1,1,-1,-1;1,-1,1,-1;1,-1,-1,1]/2$. This group is also known as the Weyl group of F_4. Let G act by linear transformations on x_1, x_2, x_3, x_4 and thus on the polynomial ring $\mathbf{Q}[x_1, x_2, x_3, x_4]$. Then it is known that the G-invariant subring of $\mathbf{Q}[x_1, x_2, x_3, x_4]$ is a polynomial ring with generators of degrees 2, 6, 8, 12, call them A_2, A_6, A_8, A_{12}. Then $\mathbf{Q}(x_1, x_2, x_3, x_4)$ is a normal extension of $\mathbf{Q}(A_2, A_6, A_8, A_{12})$ with Galois group G. For (c_1, c_2, c_3, c_4) outside the union of finitely many hyperplanes in $\mathbf{Q}^4$, this extension is generated by $X := c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4$, and X has 1152 conjugates all in a four-dimensional space over $\mathbf{Q}$. By the Hilbert irreducibility theorem there exist rational a_2, a_6, a_8, a_{12} such that when we substitute a_i for the corresponding A_i we obtain an extension of $\mathbf{Q}$ with the same Galois group G. The resulting algebraic number X satisfies the criterion of the problem.

"Indeed it is known that of the N smallest choices for (a_2, a_6, a_8, a_{12}) all but $o(N)$ work as $N \to \infty$; see for instance Chapter 3 of Serre's Topics in Galois Theory for Hilbert irreducibility and its applications to this kind of inverse Galois problem."

Remark: Much can be said about the analogous question for dimensions other than 4. The interested reader is referred to Chris Smyth for details.

Remark: (new) Chris Smyth reports that a paper arising from this problem has been mounted on arXiv, at http://arXiv.org/abs/math.NT/0308069.

001:16 (Hugh Edgar) Does $1 + q + \ldots + q^{x-1} = p^y$ have any solutions with p and q odd primes, $x > 3$ and $y > 1$ other than $(p, q, x, y) = (11, 3, 5, 2)$?

Remarks: 1. Hugh offers \$50 (U.S.) for the solution to this problem.

2. This problem appears, without the monetary offer, as D10 in UPINT.

3. Florian Luca notes that there are several recent papers on the equation $1 + q + \ldots + q^{x-1} = p^y$ by Bugeaud, Mignotte and others, e.g.,

Y. Bugeaud, G. Hanrot, M. Mignotte, Sur l'équation diophantienne $(x^{n-1})/(x-1) = y^q$, III, Proc. London Math. Soc. (3) 84 (2002) 59–78.

Remark: (new) Florian Luca also points out the paper,

T. N. Shorey, Some conjectures in the theory of exponential Diophantine equations, Publ. Math. Debrecen 56 (2000) 631–641, MR 2001i:11038.

001:22 (Gary Walsh) Is there a heuristic that suggests that $(x^3 - 1)(y^3 - 1) = z^2$ has infinitely many solutions with integers x, y, and 1 distinct?

Remark: Noam Elkies writes, "The usual heuristics suggest that there should be only finitely many solutions ...."

Remark: (new) Walsh notes that for fixed d the equation $(x^3 - 1)(y^3 - 1) = -dz^2$ may have infinitely many solutions, citing the following construction due to Frits Beukers. Fix $d > 1$ such that $u^2 - dv^2 = -1$ has a solution (hence, infinitely many solutions); then let $x = 1 + 3u^2$, $y = 1 - 3dv^2$ (note that $x + y = -1$, whence $x^2 + x + 1 = y^2 + y + 1$), and $z = 3uv(x^2 + x + 1)$.

002:01 (Carl Pomerance) Let S be an open subset of the positive reals, closed under addition, and not containing 1. Let m be the dx/x measure, that is, $m(A) = \int 1_A(x)\,\frac{dx}{x}$ where 1_A is the characteristic function of A. Is it true that $m\big(S \cap (0,t)\big) \le t$ for all t, $0 < t \le 1$? What is the right conjecture for Lebesgue measure?

Remarks (about the dx/x version): Seva Lev proves $m\big(S \cap (0,t)\big) \le 2\log(1+t)$ using work of Dixmier on an extremal version, due to Erdős and Graham, of the Frobenius postage stamp problem. Hendrik Lenstra and Peter Stevenhagen construct, for given ϵ and t, a set $S_{(\epsilon,t)}$ with $m\big(S_{(\epsilon,t)} \cap (0,t)\big) > t - \epsilon$.

Solution: Carl Pomerance reports that Daniel Bleichenbacher at Bell Labs has solved the problem, obtaining strict inequality. Also, that Lev and Bleichenbacher have the upper bound t^2 for the Lebesgue measure version of the problem (email from Seva Lev claims only a solution for $t \le t_0$, where t_0 is about 0.1), with the example $S = (1/(n+1), 1/n)$, $t = 1/n$ showing that this is essentially best possible. Pomerance continues, "Bleichenbacher's proof uses duality in linear programming. It is hoped now that Bleichenbacher's solution to the problem will lead to a version of the new primality test which can get by with less esoteric analytic tools than the theorems of Bombieri, Friedlander, and Iwaniec."

002:02 (John Jaroma) The difference equation $x_{n+1} = x_n^2 - 2$ has the general solution $x_n = c^{2^n} + c^{-2^n}$ where c is given by $x_0 = c + c^{-1}$. For what integer values of b other than $b = 0$ and $b = -2$ can one solve $x_{n+1} = x_n^2 + b$ explicitly?

Remarks: Peter Montgomery and David Terr point to the literature on the Pollard rho method of factorization, which relies on this recurrence. Montgomery also points to its relevance to the definition of the Mandelbrot set.

002:03 (Neville Robbins) Estimate $\sum \#\{\, g : g^{p-1} \equiv 1 \pmod{p^2} \,\}$ where the sum is over all primes $p \le X$ and over primitive roots $g \pmod p$ with $0 < g < p$.

Note that if p is an odd prime and g is a primitive root $\pmod p$ then it is also a primitive root $\pmod{p^n}$ for all n unless $g^{p-1} \equiv 1 \pmod{p^2}$, so we are counting the number of primitive roots $\pmod p$ in $(0,p)$ that fail to be primitive roots modulo higher powers of p.

Remark: All solutions to $g^{p-1} \equiv 1 \pmod{p^2}$, $2 \le g \le 99$, $3 \le p < 2^{32}$, are given in

P. L. Montgomery, New solutions of $a^{p-1} \equiv 1 \ (mod\ p^2)$, Math. Comp. 61 (1993) 361–363, MR 94d:11003.

002:04 (David Bailey) Is there a Bailey-Borwein-Plouffe formula for $\arctan 2/7$, that is, a formula $\arctan 2/7 = \sum_{k=0}^{\infty} \frac{p(k)}{2^k q(k)}$ where p and q are polynomials with integer coefficients, p of lower degree than q, and q has no positive integer zeros? Such formulas are known for $\arctan a/b$ for all fractions a/b "simpler" than $2/7$.

Solution: David Bailey writes that the problem has now been solved in the negative. He gives the reference,

Jonathan M. Borwein, David Borwein, William F. Galway, Finding and excluding b-ary Machin-type BBP formulae, http://www.cecm.sfu.ca/Preprints03/preprints03.html

002:05 (Hugh Edgar) Are there examples, other than $1+5+5^2 = 1+2+2^2+2^3+2^4 = 31$, of $1+p+p^2+\ldots+p^x = 1+q+q^2+\ldots+q^y = r$ with p, q, and r distinct primes?

Remark: Peter Montgomery notes $1+2+2^2+\ldots+2^{12} = 1+90+90^2 = 8191$, but of course 90 is not prime.

002:06 (Hugh Edgar) Do there exist distinct primes p and q such that $2^p \equiv 3 \pmod{q}$ and $2^q \equiv 3 \pmod{p}$?

Remark: If so, then $n = pq$ is a solution of $2^n \equiv 3 \pmod{n}$. Only three solutions to this congruence are known. A reference is Joe Crump's post of 18 September 2000 to the Number Theory list, archived at http://listserv.nodak.edu/archives/nmbrthry.html

002:07 (Kevin O'Bryant) Let $B_\alpha(k) = \#\{\, 1 \le q < k : \{q\alpha\} < \{k\alpha\}\,\}$ for α real and irrational. Let A_α be the range of B_α.

1. Is it true that A_α is never the full set of non-negative integers? It seems that $B_{\sqrt{2}}(k)$ is never 7, $B_{-\phi}(k)$ is never 3 (where ϕ is the golden ratio), and $B_{1/e}(k)$ is never 22.

2. Is it true that for each α there is a positive m such that $B_\alpha(k) = m$ infinitely often?

3. Is it true that if the continued fraction for α has bounded partial quotients then the density of A_α exists and is strictly between 0 and 1?

Solution (to question 1, by O'Bryant, Dennis Eichhorn, and Josh Cooper): For a given α, suppose that $\{q\alpha\} < \{q'\alpha\}$ are the two smallest values among $\{\alpha\}, \{2\alpha\}, \ldots, \{n\alpha\}$. Then $B(q) = 0$, $B(2q) = 1$, $\ldots$, $B(aq) = a-1$, where $a = [\{q'\alpha\}/\{q\alpha\}]$. Thus if the odd-numbered partial quotients of α are unbounded then B_α takes on every non-negative integer value infinitely often. This answers question 1 in the negative, while shedding no light on whether (for example) it is true that $B_{\sqrt{2}}(k)$ is never 7. However, $B_{1/e}(1061455212978359) = 22$.

With Simon Byrne, an undergraduate at Macquarie, your editor has proved that $B_{-\phi}(k)$ is never 3 (where ϕ is the golden ratio).

Remark: Josh Cooper, using the theory of quasirandom permutations, shows that for every α and every n every subinterval of $[0, n]$ of length $O(\sqrt{n \log n})$ contains an element of A_α. See math.ucsd.edu/~jcooper.

002:08 (Doug Iannucci) Are there infinitely many pairs of consecutive integers, each a product of two primes?

Remarks: 1. Yes, if any of a number of special cases of Schinzel's Hypothesis H is true. For example, if $2x-1$ and $3x-1$ are both prime then $6x-2=2(3x-1)$ and $6x-3=3(2x-1)$ are such a pair of consecutive positive integers.

2. Carl Pomerance suggests seeing whether Chen's Theorem is applicable.

3. David Terr refers to Carlos Rivera's website, www.primepuzzles.net, but I didn't see anything relevant there.

002:09 (David Terr) A floor exponential prime sequence (FEPS) is a sequence of primes $p_1, p_2, \ldots, p_r$ for which there is a real number $\theta > 2$ such that $p_j = [\theta^j]$ for $j = 1, 2, \ldots, r$. Are these conjectures true:

1. There is a FEPS of length r for every r.

2. There is no FEPS of infinite length.

3. Let $\Pi_r(x)$ be the number of FEPS of length r with $\theta \le x$. Then $\Pi_r(x)$ is asymptotic to $(x/\log x)^r/r!$ as x goes to infinity.

4. Let $\Pi(x) = \sum_r \Pi_r(x)$ be the number of FEPS of all lengths with $\theta \le x$. Then $\log \Pi(x)$ is asymptotic to $x/\log x$.

5. For a fixed $x \ge 2$, the distribution of lengths of FEPS with $\theta < x$ is asymptotically Poisson with mean and variance $x/\log x$.

002:10 (Kiran Kedlaya) Fix a positive integer k, and fix integers $a_1, b_1, \ldots, a_k, b_k$. Let $f_i(x) = a_i x + b_i$, $i = 1, 2, \ldots, k$. Then

$$\#\{\, x : -N \le x \le N, f_i(x) \text{ squarefree for all } i \,\} = cN + \epsilon(N)$$

where $c = \prod_p \frac{1}{p^2} \#\{\, x : 0 \le x \le p^2 - 1, f_i(x) \not\equiv 0 \pmod{p^2} \text{ for all } i \,\}$ (the product is over all primes) and $\epsilon(N) = o(N)$. See

George Greaves, Power-free values of binary forms, Quart. J. Math. 43 (1992) 45–65, MR 92m:11098,

C. Hooley, On the power free values of polynomials, Mathematika 14 (1967) 21–26, MR 35 #5405.

Is there an explicit bound, using the methods of these papers or otherwise, for $|\epsilon(N)|$ in terms of $N, a_1, b_1, \ldots, a_k, b_k$, and c?

002:11 (Leanne Robertson) Are these conjectures true:

1. Any tree with n vertices can be labeled with the integers $1, 2, \ldots, n$ in such a way that adjacent vertices have coprime labels.

2. Any tree with n vertices, $n < 17$, can be labeled with any n consecutive integers in such a way that adjacent vertices have coprime labels.

Remarks: 1. It is known that the first conjecture is true for $n \le 94$.

2. If true, then the second conjecture is best possible. None of the 17 consecutive numbers $2184, 2185, \ldots, 2200$ is relatively prime to all 16 of the others, so the "star" consisting of one vertex joined to each of 16 others can't have a coprime labelling with

these numbers. Deciding the second conjecture is a finite problem, as there are only finitely many trees on 16 or fewer vertices, and only finitely many "coprimeness patterns" among sets of 16 or fewer consecutive integers.

002:12 (Doug Iannucci) Let D be the multiplicative function on the positive integers satisfying $D(p^a) = ap^{a-1}$ for all primes p.

1. Is the sequence $n, D(n), D(D(n)), \ldots$ bounded for all n?
2. Does any such sequence of iterates lead to a cycle of length 7?

Remarks: The sequence beginning with $n = 31^{124}$ has been pursued to 48 million iterations without the appearance of a cycle. Also, no cycle has been found for $n = 23^{92}$. Cycles of length k are known only for $k = 8$ and $1 \le k \le 6$.

002:13 (Lenny Jones) A finite group G is a perfect order subsets (POS) group if for every d such that G contains an element of order d the number of elements of order d in G divides the order of G.

1. Let G be an abelian POS whose order is not a power of 2. Must 3 divide the order of G?
2. Are there any simple non-abelian POS groups?
3. An abelian POS group $(\mathbf{Z}_2)^t \times M$ with M of odd order is said to be minimal if there is no proper subgroup M' of M such that $(\mathbf{Z}_2)^t \times M'$ is a POS group. Are there any minimal POS groups other than $(\mathbf{Z}_2)^{11} \times \mathbf{Z}_3 \times \mathbf{Z}_5 \times (\mathbf{Z}_{11})^2 \times \mathbf{Z}_{23} \times \mathbf{Z}_{89}$ that have a non-cyclic Sylow p-subgroup for an odd prime p?

Remark: POS groups are discussed in

Carrie E. Finch, Lenny Jones, A curious connection between Fermat numbers and finite groups, Amer. Math. Monthly 109 (2002) 517–524.

002:14 (Lenny Jones) Does $\sigma\left(\prod_{i=1}^k p_i^{a_i}\right) = \prod_{i=1}^k p_i$, $a_i \ge 0$ have infinitely many solutions in primes $p_1, \ldots, p_k$? One solution is $\sigma(3^2 \times 13^2 \times 61^0) = 3 \times 13 \times 61$.

002:15 (Lenny Jones) Given primes $p_1 < p_2 < \ldots < p_{k-1}$ such that $p_i + 1$ divides $p_1^2 p_2^2 \times \ldots \times p_{i-1}^2$ for $i = 2, \ldots, k-1$, does there always exist a prime $p_k > p_{k-1}$ such that $p_k + 1$ divides $p_1^2 p_2^2 \times \ldots \times p_{k-1}^2$?

002:16 (Hugh Edgar) Minimize K such that $\sigma(n) \le H_n + Ke^{H_n} \log H_n$ for all integer $n \ge 1$. Here $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$.

Remark: Greg Martin has achieved $K = 1.0254$. See also

Jeffrey C. Lagarias, An elementary problem equivalent to the Riemann Hypothesis, Amer. Math. Monthly 109 (2002) 534–543.

002:17 (Hugh Edgar) Find all integral solutions of $x^3 + y^3 + z^3 = 3$.

Remark. The solutions $(1, 1, 1)$ and $(4, 4, -5)$ are known.

002:18 (Neville Robbins) For p prime, let $f(p) = \frac{p-1}{2} - \phi(p-1)$, so $f(p)$ is the number of quadratic non-residues that aren't primitive roots. Are there infinitely many positive integers r such that $f(p) = r$ has no solution?

002:19 (Peter Montgomery) As Bruce Reznick pointed out, if $x_1, \ldots, x_4$ are complex numbers and $\sum x_i = \sum x_i^2 = 0$ then $\sum x_i^5 = 0$. What can be said about non-commutative rings in which the same conclusion holds?

Remark: Montgomery shows that there are counterexamples in the quaternions, e.g., $\sum x_i = \sum x_i^2 = 0 \neq \sum x_i^5$ if $x_1 = 3$, $x_2 = 1+\sqrt{2}(i-j)$, $x_3 = 1+\sqrt{2}(j-k)$, $x_4 = 1+\sqrt{2}(k-i)$. He also shows that in any ring that contains the field of 8 elements there is a counterexample based on the four roots of $x^4 + x^2 + x = 0$ in the field.

002:20 (David Moulton) We consider sequences $b_0, b_1, \ldots$ of positive integers such that every positive integer can be written as $\sum^r a_j b_j$ for some r and some non-negative integers $a_0, \ldots, a_r$. We consider the effect that conditions on the coefficients $a_0, a_1, \ldots$ have on the allowable sequences $b_0, b_1, \ldots$. The conditions on coefficients we consider are given as n-tuples of non-negative integers; the condition $(c_1, \ldots, c_s)$ means that for all i it is forbidden to have $a_i \geq c_1$ and $\ldots$ and $a_{i+s-1} \geq c_s$. A sequence may be subject to one or more such conditions.

1. For which sequences and conditions will every positive integer have a unique representation?

The greedy sequence corresponding to a given set of conditions is the sequence in which, having chosen $b_0, b_1, \ldots, b_{i-1}$, we choose b_i as large as possible.

2. For which sets of conditions does the greedy sequence satisfy a (constant-coefficient, linear) recurrence? How can we determine the recurrence, if any, from the conditions? Does the greedy sequence for $\{(4), (1,1)\}$ satisfy a recurrence?

3. Is there a sequence such that every positive integer has a unique representation but the sequence satisfies no recurrence?

4. What is the fastest growth rate possible for a sequence subject to given conditions?

5. Under what circumstances can a non-greedy sequence for a given set of conditions grow faster than the greedy sequence for those conditions?

6. Given two sets of conditions, one a proper subset of the other, is it possible that the greedy sequence for the larger set grows faster than that for the smaller set?

Remarks: 1. Here are some illustrations. The greedy sequence for $\{(3), (1,1)\}$ is the Lucas sequence $1, 3, 4, 7, 11, 18, \ldots$. The representations are not unique. For example, $6 = 2b_0 + b_2 = 2b_1$. The greedy sequence for $\{(3), (1,1), (0,2)\}$ is also the Lucas sequence, but now the representations are unique (e.g., $6 = 2b_1$ is no longer permitted). The conditions $\{(3), (1,1)\}$ also permit the non-greedy sequence $1, b_1, 3, b_3, 9, b_5, 27, \ldots$, where $b_1, b_3, b_5, \ldots$ are arbitrary, and this sequence grows faster than the Lucas sequence.

The greedy sequence for $\{(4), (1,1)\}$ begins $1, 4, 5, 9, 14, 23, 37$, which satisfies the recurrence $a_n = a_{n-1} + a_{n-2}$, but then $b_7 = 148$.

2. Kevin O'Bryant writes that

Aviezri S. Fraenkel, Systems of numeration, Amer. Math. Monthly 92 (1985) 105–114, MR 86d:11016

deals with some of these questions.

002:21 (Gary Walsh) Are there any solutions to $x^2-2=p^n$ with p prime, x and n integers, $n>1$?

Remarks: This is the one case of $x^2=p^a \pm p^b+1$ not solved in Florian Luca's preprint about these equations. Luca notes that there are only finitely many solutions and that using linear forms in logs one can compute a bound for x. He refers to

T. N. Shorey, R. Tijdeman, Exponential Diophantine Equations, Camb. U. Pr. 1986, MR 88h:11002.

The equation is also a special case of $x^n - y^m = 2$, the equation "one up" from Catalan's. It is related to Ramanujan-Nagell type equations, and to the Bennett-Skinner equation $x^n + 2^\alpha y^n = z^2$, $\alpha > 1$.

002:22 (John Selfridge) The number 82818079 ... 1110987654321 (obtained by concatenating the numbers $82, 81, \ldots, 1$) is prime. Is there any other n for which the number obtained by concatenating $n, n-1, \ldots, 1$ is prime? Is there any n for which concatenating the numbers $1, 2, \ldots, n$ gives a prime?

Remark: In

Ralf Stephan, Factors and primes in two Smarandache sequences, Smarandache Notions J. 9 (1998) 4–10,

available at http://me.in-berlin.de/~rws, it is claimed that the given prime is the only one for n up to 750, and that there are no primes in the other sequence for n up to 840.

002:23 (Josh Cooper) Given a vector $\mathbf{a} = (a_1, \ldots, a_n)$ of distinct positive integers with no common factor, let $\mathcal{F} = \mathcal{F}_\mathbf{a}$ be the set of positive integers that can be expressed as a non-negative integer linear combination of $a_1, \ldots, a_n$, and let $N = N_\mathbf{a}$ be the largest integer not in $\mathcal{F}$. For $0 < \beta < 1$ does $\lim_{a_1,\ldots,a_n\to\infty} \frac{1}{N}\#\{\,\mathcal{F} \cap [0, \beta N]\,\}$ exist and, if so, what is it?

Remarks: 1. For $n = 2$ it is well-known that $N = a_1 a_2 - a_1 - a_2$, and Cooper proves that the limit in question exists and is $\beta^2/2$.

2. Seva Lev has worked out several cases where the computations can be carried out explicitly. For example, $\mathbf{a} = (m+1, \ldots, m+n)$, n fixed, $m \to \infty$; also $\mathbf{a} = (a_1, \ldots, a_n)$ with $a_3, \ldots, a_n > N_{(a_1,a_2)}$. In all these cases,

$$\frac{1}{N}\#\{\,\mathcal{F}_\mathbf{a} \cap [1, \beta N]\,\} = \frac{1}{2}\beta^2 + o(1).$$

Thus if the limit exists it is $\beta^2/2$.

002:24 (John Brillhart) Can the primitive part of $b^n - 1$, or any factor of the primitive part of $b^n - 1$, be a Carmichael number?